COMP 355
Advanced Algorithms

Dynamic Programming:
Space Efficient Sequence Alignment
Section 6.7(KT)

Rhodes College
—1848—

Edit Distance

Applications.

- Basis for Unix diff.
- Speech recognition.
- Computational biology.

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]

- Gap penalty δ; mismatch penalty α_{pq}.
- Cost = sum of gap and mismatch penalties.

C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	T	A	C	A	T

$\alpha_{TC} + \alpha_{GT} + \alpha_{AG} + 2\alpha_{CA}$

-	C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	-	T	A	C	A	T

$2\delta + \alpha_{CA}$

Sequence Alignment: Problem Structure

Def. OPT(i, j) = min cost of aligning strings $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$.

- Case 1: OPT matches x_i-y_j.
  - pay mismatch for x_i-y_j + min cost of aligning two strings $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_{j-1}$
- Case 2a: OPT leaves x_i unmatched.
  - pay gap for x_i and min cost of aligning $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_j$
- Case 2b: OPT leaves y_j unmatched.
  - pay gap for y_j and min cost of aligning $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_{j-1}$

$$OPT(i, j) = \begin{cases} j\delta & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + OPT(i-1, j-1) \\ \delta + OPT(i-1, j) \\ \delta + OPT(i, j-1) \end{cases} & \text{otherwise} \\ i\delta & \text{if } j = 0 \end{cases}$$

Sequence Alignment: Algorithm

```
Sequence-Alignment(m, n, x₁x₂...xₘ, y₁y₂...yₙ, δ, α) {
   for i = 0 to m
      M[0, i] = iδ
   for j = 0 to n
      M[j, 0] = jδ

   for i = 1 to m
      for j = 1 to n
         M[i, j] = min(α[xᵢ, yⱼ] + M[i-1, j-1],
                       δ + M[i-1, j],
                       δ + M[i, j-1])
   return M[m, n]
}
```

- Analysis. $\Theta(mn)$ time and space.
- English words or sentences: $m, n \leq 10$.
- Computational biology: $m = n = 100{,}000$. 10 billions ops OK, but 10GB array?

Sequence Alignment: Linear Space

Q. Can we avoid using quadratic space?

Easy. Optimal value in O(m + n) space and O(mn) time.

- Compute OPT(i, •) from OPT(i-1, •).
- No longer a simple way to recover alignment itself.

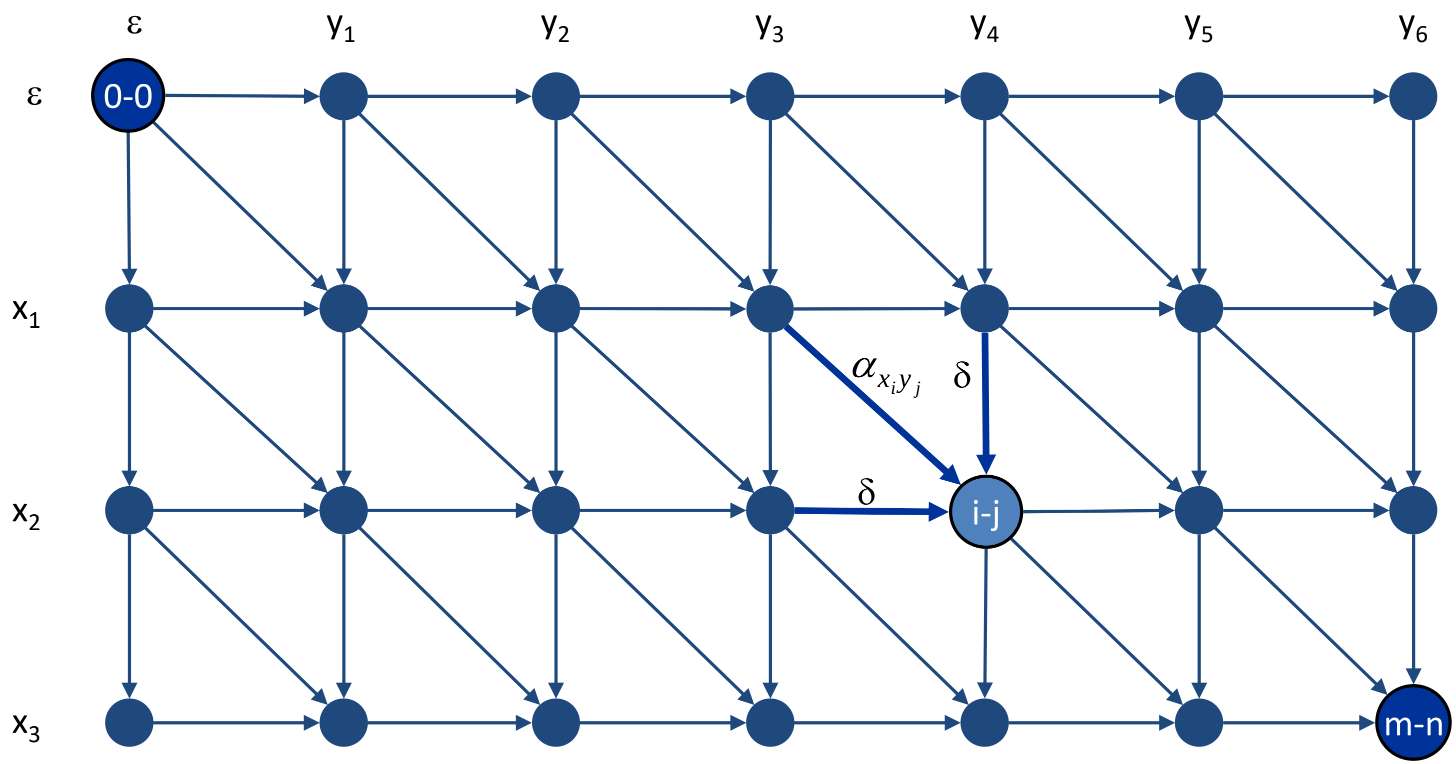
Theorem. [Hirschberg 1975] Optimal alignment in $O(m + n)$ space and $O(mn)$ time.

- Clever combination of divide-and-conquer and dynamic programming.
- Inspired by idea of Savitch from complexity theory.

Sequence Alignment: Linear Space

Edit distance graph.

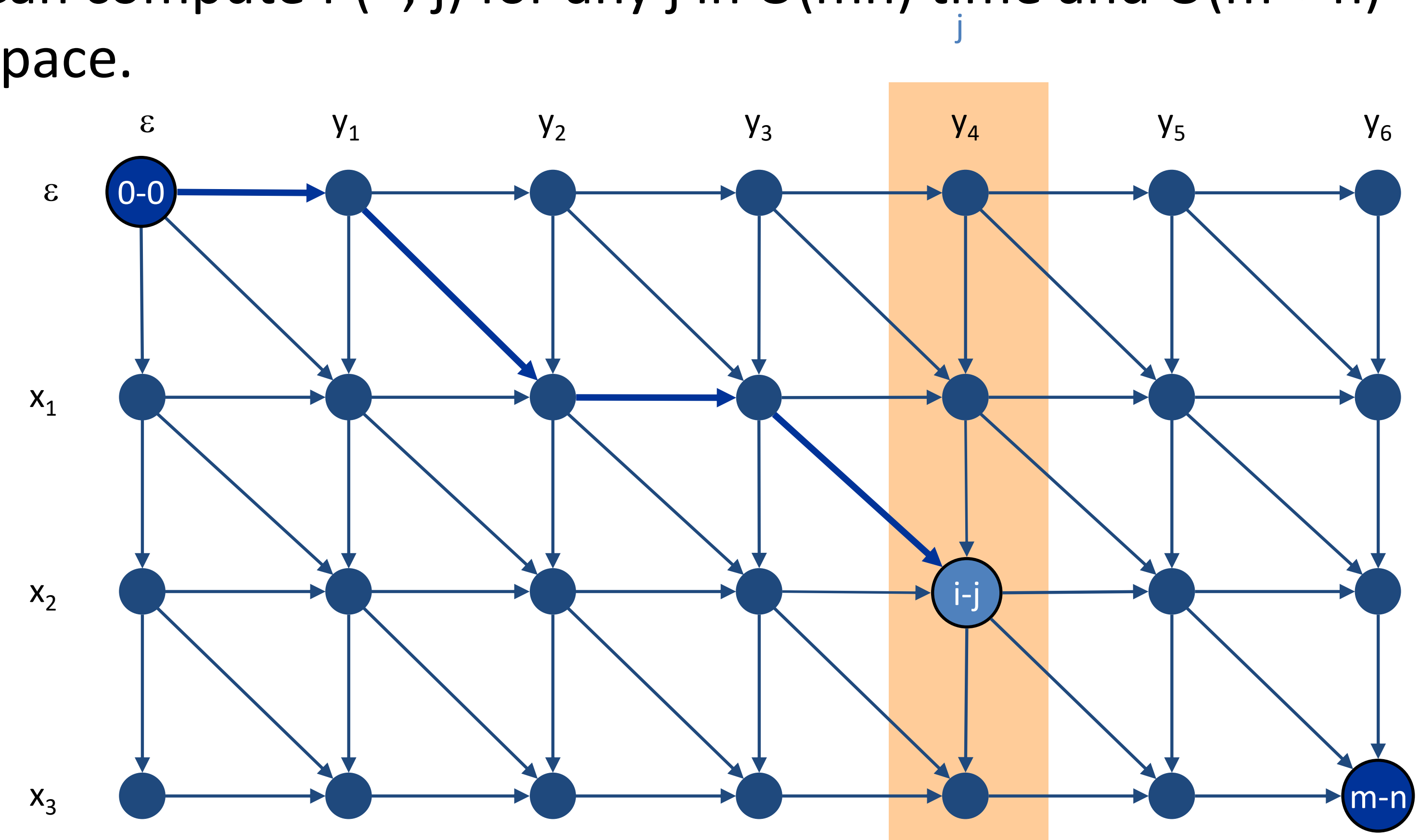
- Let f(i, j) be shortest path from (0,0) to (i, j).
- Observation: f(i, j) = OPT(i, j).



Sequence Alignment: Linear Space

Edit distance graph.

- Let f(i, j) be shortest path from (0,0) to (i, j).
- Can compute f (•, j) for any j in O(mn) time and O(m + n) space.



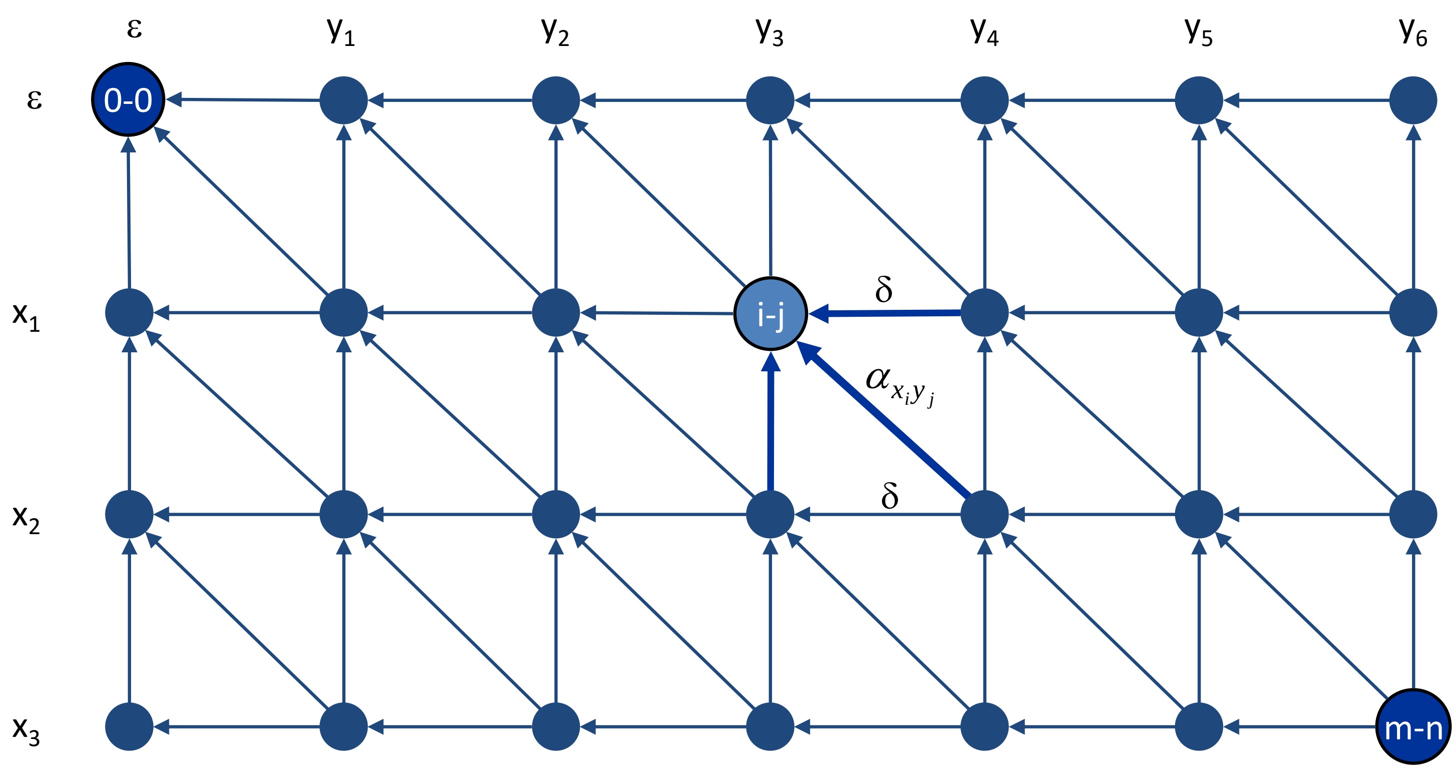
Space-Efficient Alignment Algorithm

```
Space-Efficient-Alignment(X,Y)
  Array B[0 ... m, 0 ... 1]
  Initialize B[i,0] = i δ (just as in column 0 of A)
  For j = 1, ..., n
     B[0,1]=jδ (since this corresponds to entry A[0,j])
     For i = 1, ..., m
        B[i,1] = min[α_{x_i y_j} + B[i−1,0],
                     δ + B[i−1,1], δ + B[i,0]].
     Endfor
     Move column 1 of B to column 0 to make room for next iteration:
        Update B[i,0] = B[i,1] for each i
  Endfor
```

Sequence Alignment: Linear Space

Edit distance graph.

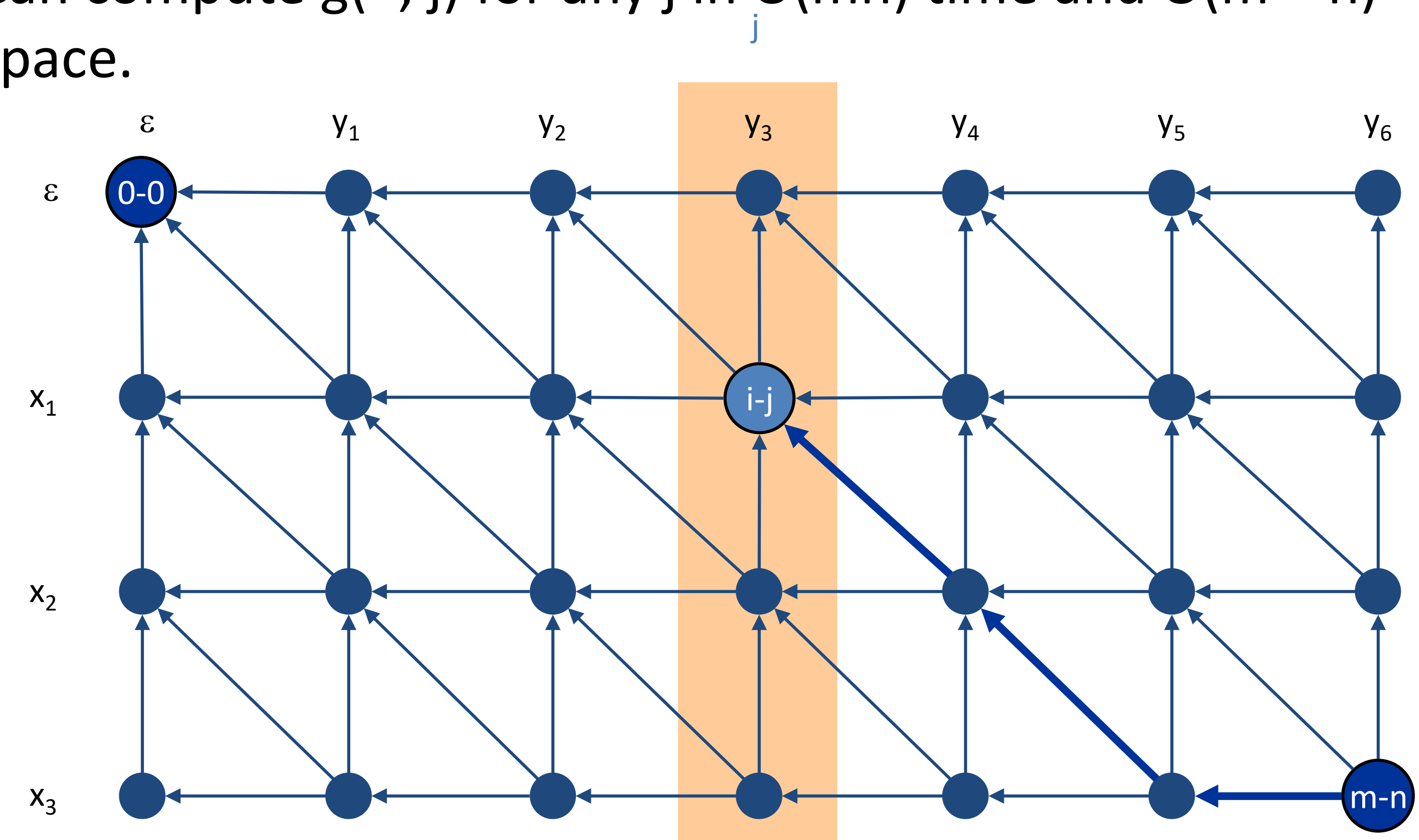
- Let g(i, j) be shortest path from (i, j) to (m, n).
- Can compute by reversing the edge orientations and inverting the roles of (0, 0) and (m, n)



Sequence Alignment: Linear Space

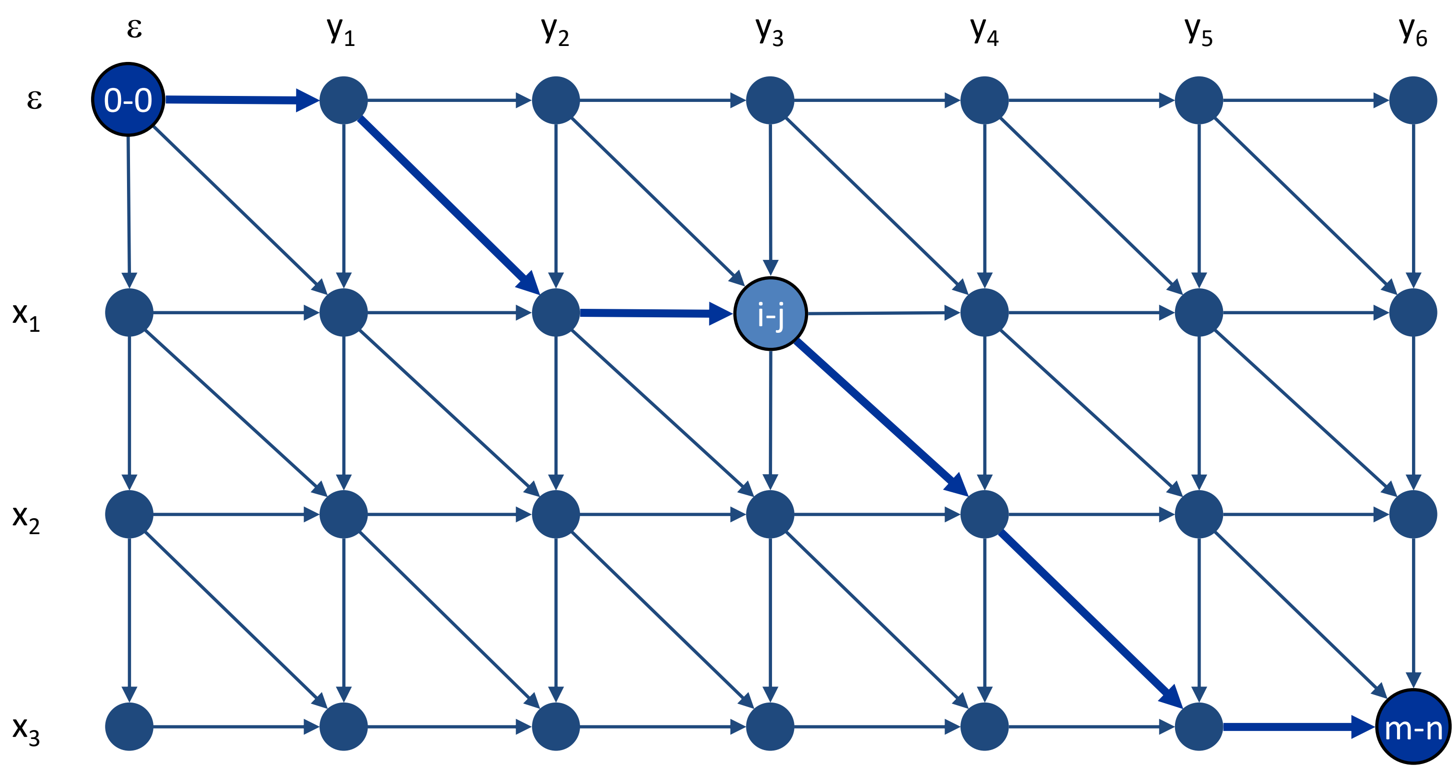
Edit distance graph.

- Let g(i, j) be shortest path from (i, j) to (m, n).
- Can compute g(•, j) for any j in O(mn) time and O(m + n) space.



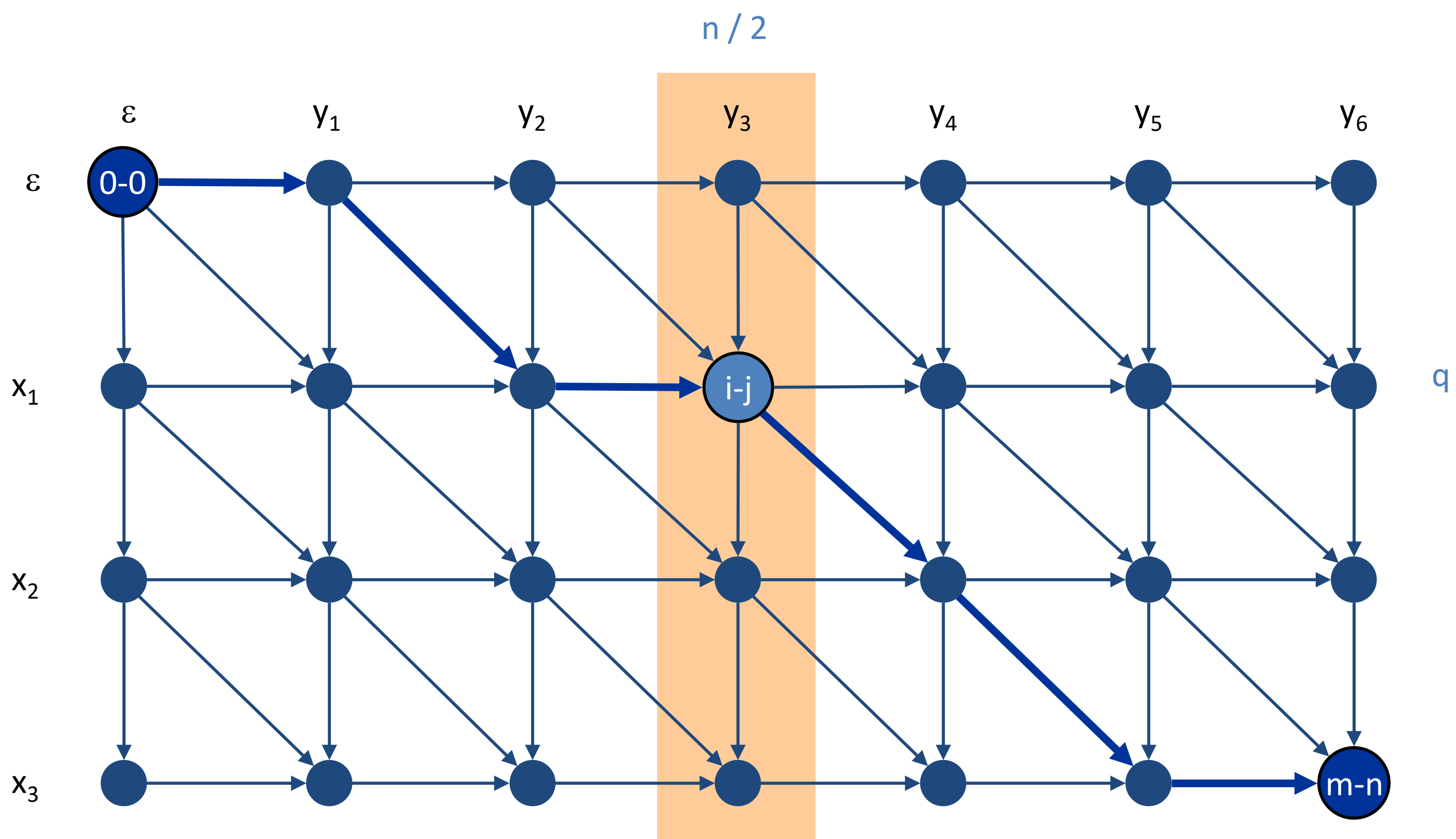
Sequence Alignment: Linear Space

Observation 1. The cost of the shortest path that uses (i, j) is $f(i, j) + g(i, j)$.



Sequence Alignment: Linear Space

Observation 2. let q be an index that minimizes f(q, n/2) + g(q, n/2). Then, the shortest path from (0, 0) to (m, n) uses (q, n/2).

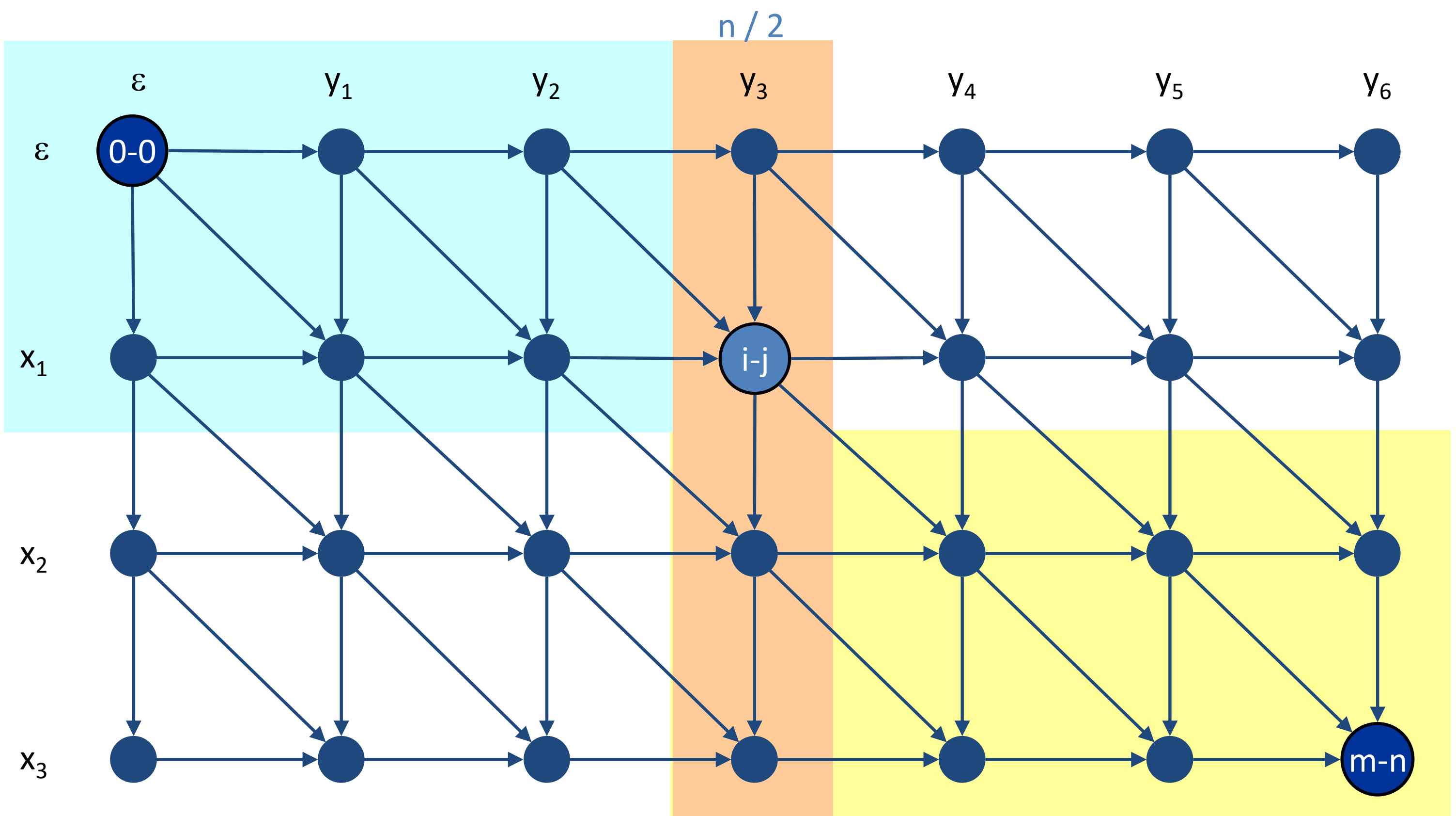


Sequence Alignment: Linear Space

Divide: find index q that minimizes f(q, n/2) + g(q, n/2) using DP.

- Align x_q and $y_{n/2}$.

Conquer: recursively compute optimal alignment in each piece.



Sequence Alignment: Running Time Analysis Warmup

Theorem. Let $T(m,n)$ = max running time of algorithm on strings of length at most m and n. $T(m,n) = O(mn \log n)$.

$$T(m,n) \leq 2T(m, n/2) + O(mn) \Rightarrow T(m,n) = O(mn \log n)$$

Remark. Analysis is not tight because two sub-problems are of size (q, n/2) and (m - q, n/2). In next slide, we save $\log n$ factor.

Sequence Alignment: Running Time Analysis

Theorem. Let $T(m,n)$ = max running time of algorithm on strings of length m and n. $T(m,n) = O(mn)$.

Pf. (by induction on n)

- $O(mn)$ time to compute $f(\bullet, n/2)$ and $g(\bullet, n/2)$ and find index q.
- $T(q, n/2) + T(m - q, n/$
- Choose constant c so that:

$$\begin{aligned} T(m,\ 2) &\leq cm \\ T(2,\ n) &\leq cn \\ T(m,\ n) &\leq cmn + T(q,\ n/2) + T(m-q,\ n/2) \end{aligned}$$

- Basis cases: $m = 2$ or $n = 2$.
- Inductive hypothesis: $T(m,n$ ... $2cmn$

$$\begin{aligned} T(m,n) &\leq T(q,n/2) + T(m-q,n/2) + cmn \\ &\leq 2cqn/2 + 2c(m-q)n/2 + cmn \\ &= cqn + cmn - cqn + cmn \\ &= 2cmn \end{aligned}$$

Divide-and-Conquer Alignment Algorithm

```
Divide-and-Conquer-Alignment(X,Y)
  Let m be the number of symbols in X
  Let n be the number of symbols in Y
  If m ≤ 2 or n ≤ 2 then
     Compute optimal alignment using Alignment(X,Y)
  Call Space-Efficient-Alignment(X,Y[1:n/2]),
     obtaining array B
  Call Backward-Space-Efficient-Alignment(X,Y[n/2+1:n]),
     obtaining array B'
  Let q be the index minimizing B[q,1] + B'[q,n]
  Add (q,n/2) to global list P
  Divide-and-Conquer-Alignment(X[1:q],Y[1:n/2])
  Divide-and-Conquer-Alignment(X[q+1:n],Y[n/2+1:n])
  Return P
```