COMP 355
Advanced Algorithms

Cook's Theorem,
3SAT, and Independent Set
Chapter 8 (KT)
Section 34.4-34.5(CLRS)

Rhodes College

—1848—

Recap

Polynomial reduction: $L_1 \leq_P L_2$ means that there is a polynomial time computable function f such that $x \in L_1$ if and only if $f(x) \in L_2$. A more intuitive way to think about this is that if we had a subroutine to solve L_2 in polynomial time, then we could use it to solve L_1 in polynomial time. Polynomial reductions are transitive, that is, $L_1 \leq_P L_2$ and $L_2 \leq_P L_3$ implies $L_1 \leq_P L_3$.

NP-Hard: L is NP-hard if for all $L' \in NP$, $L' \leq_P L$. By transitivity of $\leq_P$, we can say that L is NP-hard if $L' \leq_P L$ for some known NP-hard problem L'.

NP-Complete: L is NP-complete if (1) $L \in NP$ and (2) L is NP-hard.

It follows from these definitions that:

- If any NP-hard problems is solvable in polynomial time, then every NP-complete problem (in fact, every problem in NP) is also solvable in polynomial time.
- If any NP-complete problem cannot be solved in polynomial time, then every NP-complete problem (in fact, every NP-hard problem) cannot be solved in polynomial time.

Thus all NP-complete problems are equivalent to one another (in that they are either all solvable in polynomial time, or none are).

Satisfiability

Literal: A Boolean variable or its negation. x_i or $\overline{x_i}$

Clause: A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$

Conjunctive normal form: A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

SAT: Given CNF formula Φ, does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

Ex: $\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_2 \vee x_3\right) \wedge \left(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}\right)$

Yes: x_1 = true, x_2 = true x_3 = false.

Cook's Theorem Justification

Cook's Theorem: SAT is NP-complete.

SAT is in NP: We non-deterministically guess truth values to the variables. We can then plug the values into the formula and evaluate it. Clearly, this can be done in polynomial time.

SAT is NP-Hard:

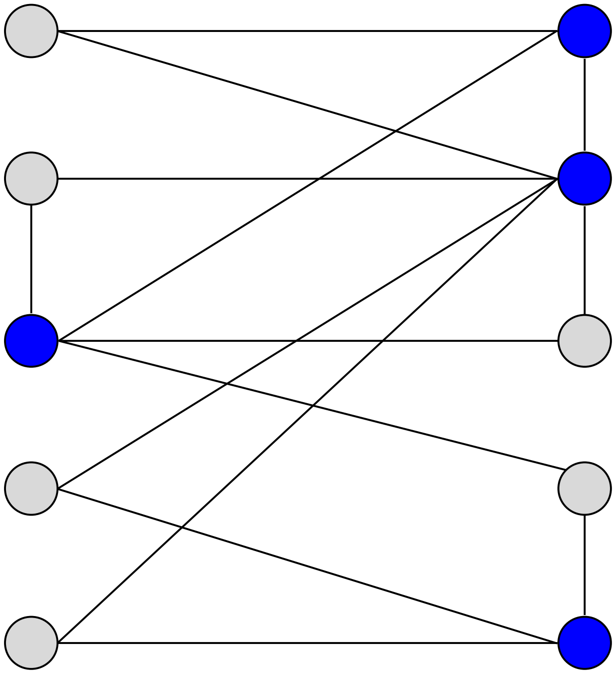
1. Every NP-problem can be encoded as a program that runs in polynomial time on a given input, subject to a number of nondeterministic guesses.
2. We can express its execution on a specific input as straight-line program that contains a polynomial number of lines of code.
3. Compile each line of code into machine code, and convert each machine code instruction into an equivalent boolean circuit
4. Express each of these circuits equivalently as a boolean formula.

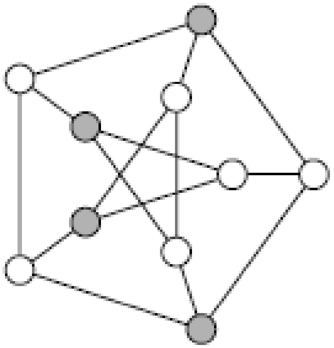
Independent Set

INDEPENDENT SET: Given an undirected graph G = (V, E) and an integer k, is there a subset of vertices S ⊆ V such that |S| ≥ k, and for each edge at most one of its endpoints is in S?

Ex. Is there an independent set of size ≥ 6? Yes.

Ex. Is there an independent set of size ≥ 7? No.





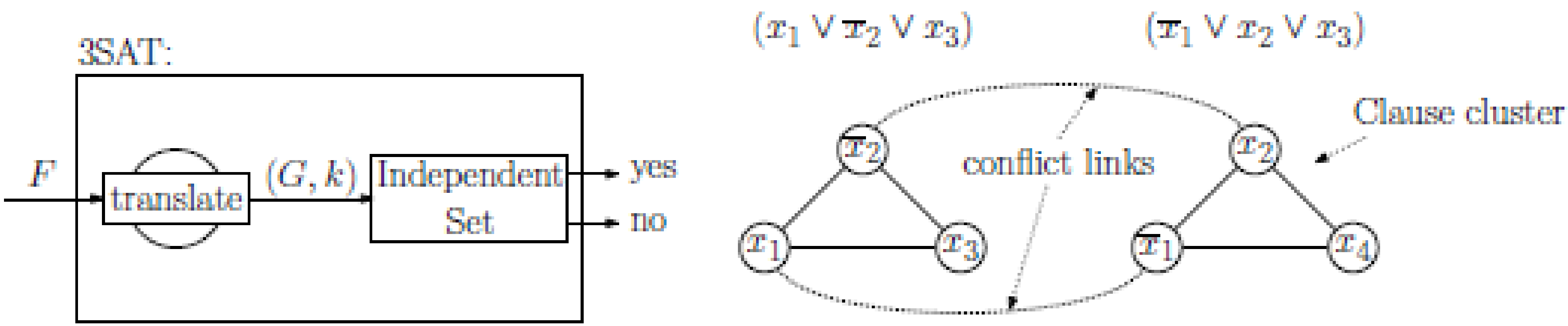
A graph with an independent set of size $k = 4$.

independent set

NP-Completeness Proof

Claim: IS is NP-complete.

1. IS ∈ NP
2. IS is NP-Hard



(a) (b)

(a) Reduction of 3-SAT to IS
(b) Clause clusters for the clauses $(x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$.

3 Satisfiability Reduces to Independent Set

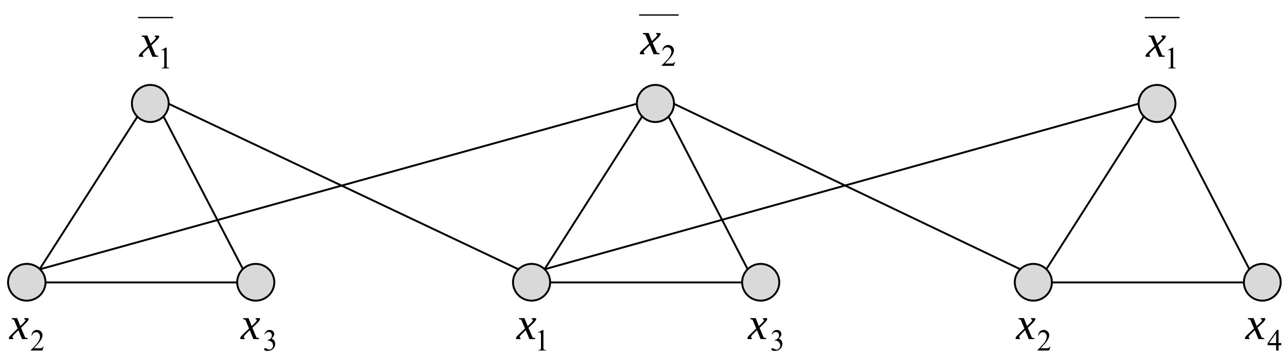
Claim. 3-SAT $\leq_P$ INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

- Set k = # of clauses in Φ
- G contains 3 vertices for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3 Satisfiability Reduces to Independent Set

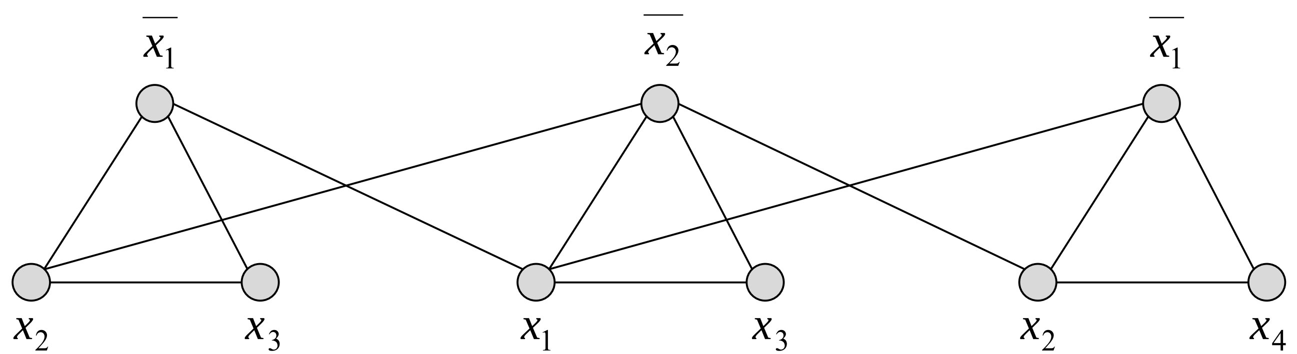
Claim. G contains independent set of size k = $|\Phi|$ iff Φ is satisfiable.

Pf. $\Rightarrow$ Let S be independent set of size k.

- S must contain exactly one vertex in each triangle.
- Set these literals to true. $\leftarrow$ and any other variables in a consistent way
- Truth assignment is consistent and all clauses are satisfied.

Pf $\Leftarrow$ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k. ▪

G



k = 3

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

3SAT to IS reduction

$$\Phi = \left(x_1 \vee \overline{x_2} \vee \overline{x_3}\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee \overline{x}_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right)$$

Reduce this 3SAT to IS

What does k need to be?

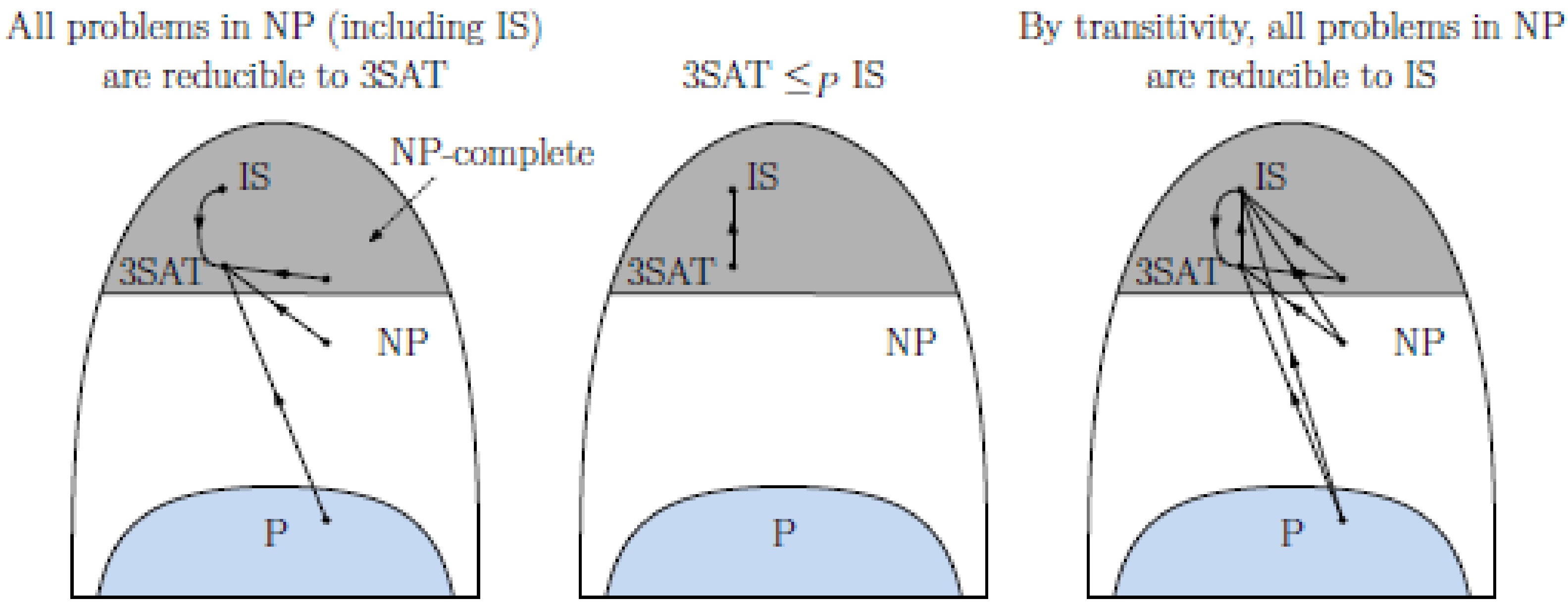
What does the graph look like?

A few things about this reduction

- Every NP-complete problem has three similar elements:
  a) something is being selected
  b) something is forcing us to select a sufficient number of such things (requirements)
  c) something is limiting our ability to select these things (restrictions).

  A reduction's job is to determine how to map these similar elements to each other.
- Our reduction did not attempt to solve the 3SAT problem.
- Remember this rule! If your reduction treats some entities different than others, based on what you think the final answer may be, you are very likely making a mistake.
- Remember, these problems are NP-complete!

Updated Picture of NP-Completeness



- By Cook's Theorem, we know that every problem in NP is reducible to 3SAT
- When we showed that IS ∈ NP, it followed immediately that IS $\leq_P$ 3SAT.
- When we showed that 3SAT $\leq_P$ IS, we established their equivalence .
- By transitivity, it follows that all problems in NP are now reducible to IS

Practice

(CLRS 34.5-8) In the **half 3-SAT** problem, we are given a 3-SAT formula F with **n** variables and **m** clauses, where m is even. We wish to determine whether there exists a truth assignment to the variables of F such that exactly half the clauses evaluate to False (0) and exactly half the clauses evaluate to True (1).

Prove that the half 3-SAT problem is NP-complete.

1. Half 3-SAT $\in$ NP
2. Half 3-SAT $\in$ NP-Hard
   Use: 3-SAT $\leq_P$ Half 3-SAT