COMP 355
Advanced Algorithms

Independent Set, Clique, and Vertex Cover

Chapter 8 (KT)

Section 34.5(CLRS)

Rhodes College
—1848—

Practice

(CLRS 34.5-8) In the **half 3-SAT** problem, we are given a 3-SAT formula f with **n** variables and **m** clauses, where m is even. We wish to determine whether there exists a truth assignment to the variables of f such that exactly half the clauses evaluate to False (0) and exactly half the clauses evaluate to True (1).

Prove that the half 3-SAT problem is NP-complete.

1. Half 3-SAT $\in$ NP
2. Half 3-SAT $\in$ NP-Hard
   Use: 3-SAT $\leq_P$ Half 3-SAT

Some NP-Complete Problems

Clique (CLIQUE): Given an undirected graph G = (V, E) and an integer *k*, does G have a subset V′ of k vertices such that for each distinct u, v ∈ V′, (u, v) ∈ E.

- Does G have a *k* vertex subset whose induced subgraph is complete (a clique)?

Vertex Cover (VC): A vertex cover in an undirected graph G = (V,E) is a subset of vertices V′ ⊆ V such that every edge in G has at least one endpoint in V′.

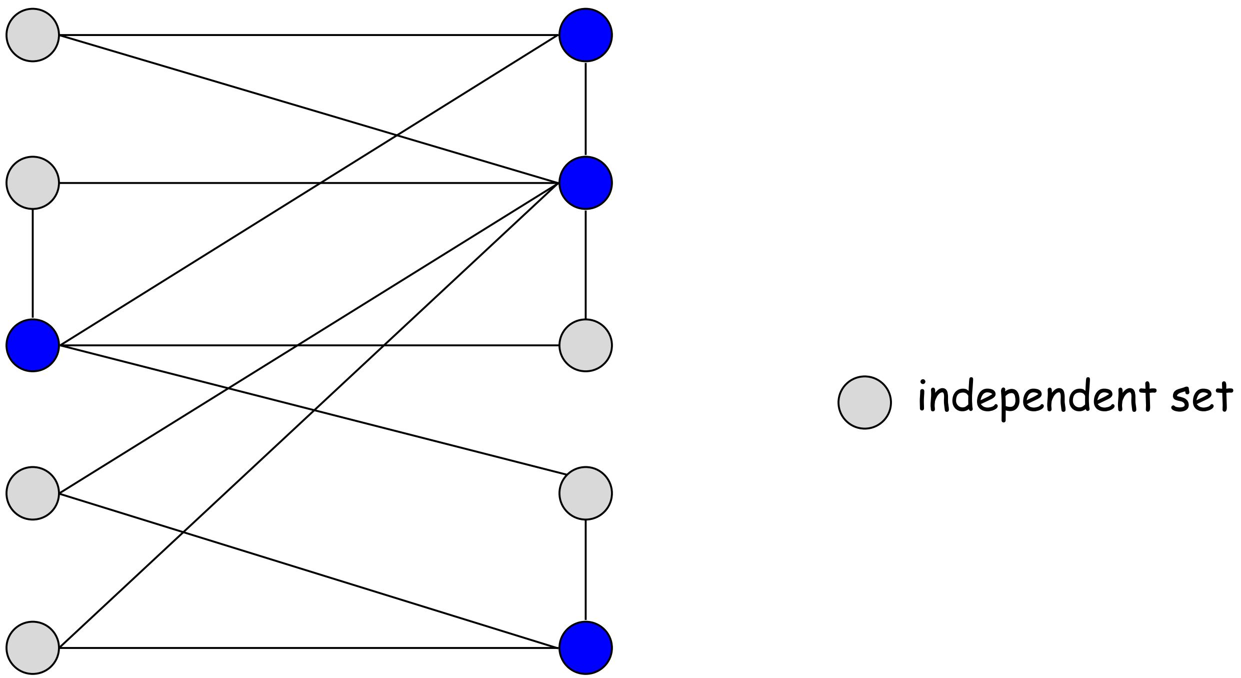
- Given an undirected graph G and an integer *k*, does G have a vertex cover of size *k*?

[Recall] Independent Set

INDEPENDENT SET: Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?

Ex. Is there an independent set of size ≥ 6? Yes.

Ex. Is there an independent set of size ≥ 7? No.

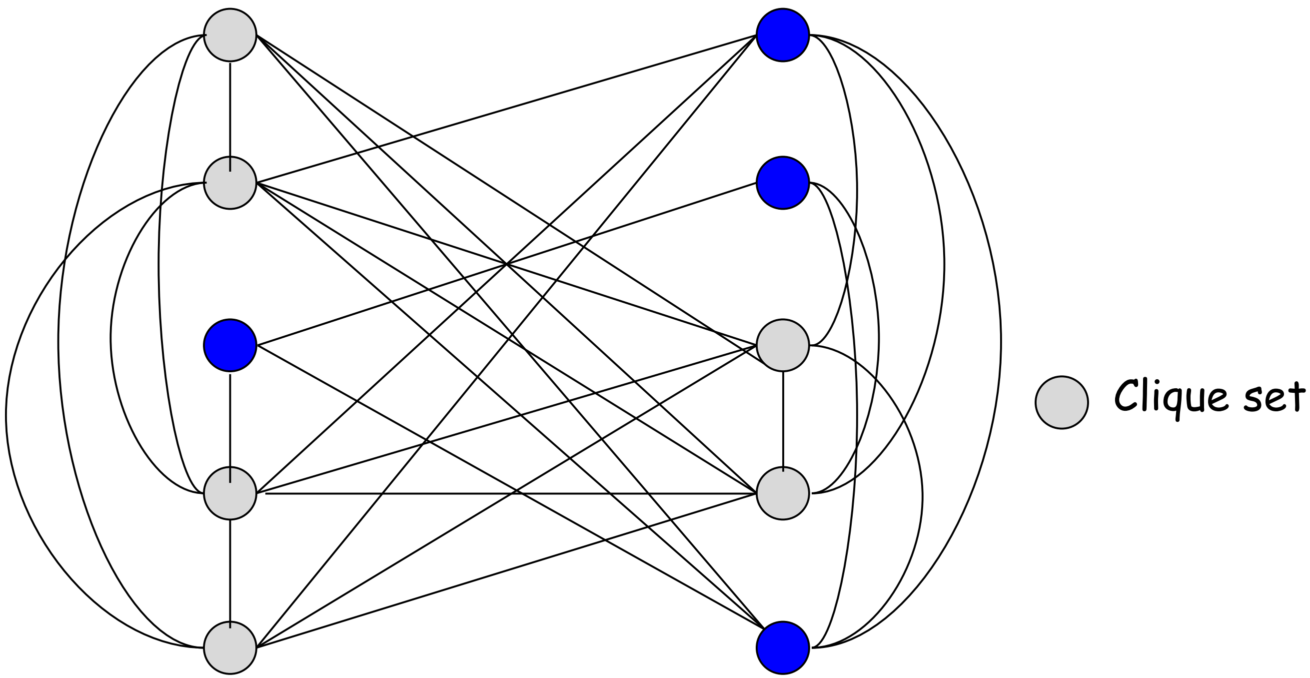


Clique

CLIQUE: Given an undirected graph G = (V, E) and an integer *k*, does G have a subset V′ of k vertices such that for each distinct u, v ∈ V′, (u, v) ∈ E.

Ex. Is there an clique set of size ≥ 6? Yes.

Ex. Is there an clique set of size ≥ 7? No.



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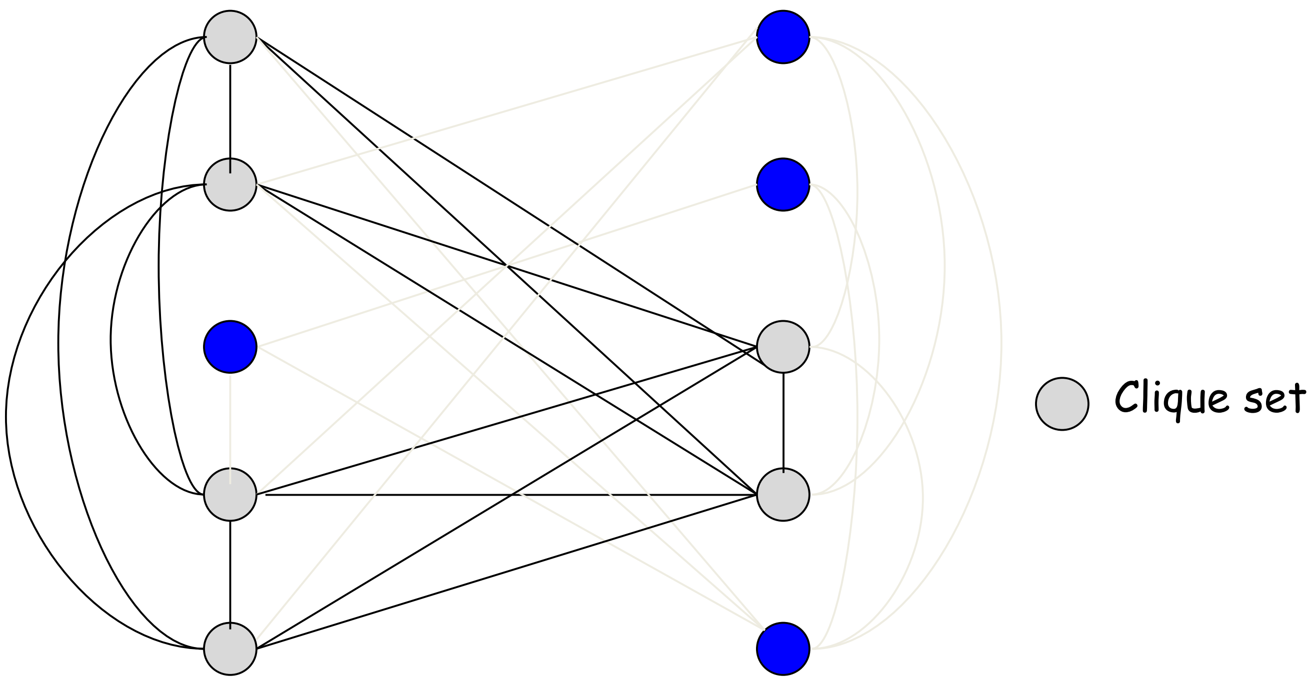
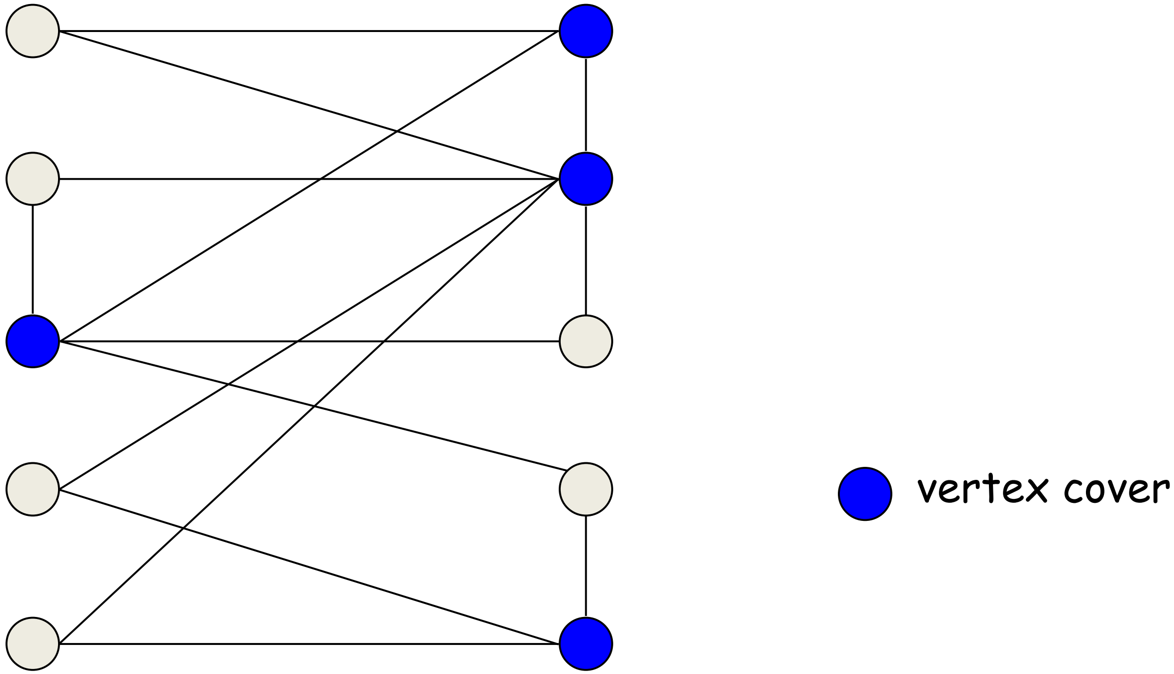


Vertex Cover

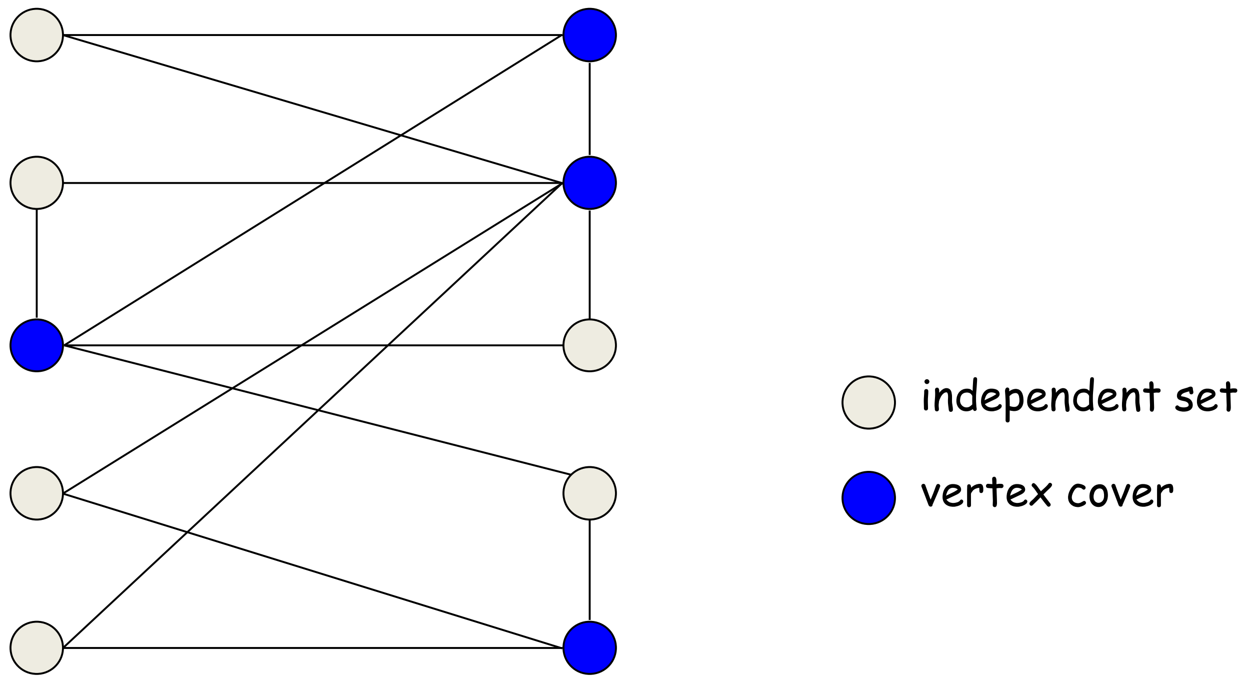
VERTEX COVER: Given a graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

- Ex. Is there a vertex cover of size ≤ 4? Yes.
- Ex. Is there a vertex cover of size ≤ 3? No.

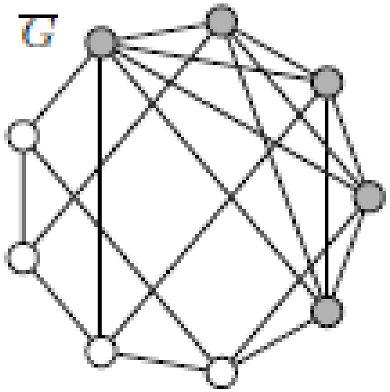


Vertex Cover and Independent Set

- Claim. VERTEX-COVER $\equiv_P$ INDEPENDENT-SET.
- Pf. We show S is an independent set iff V – S is a vertex cover.

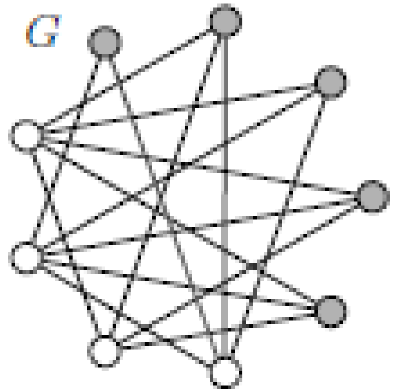


Clique, Independent set, and Vertex Cover



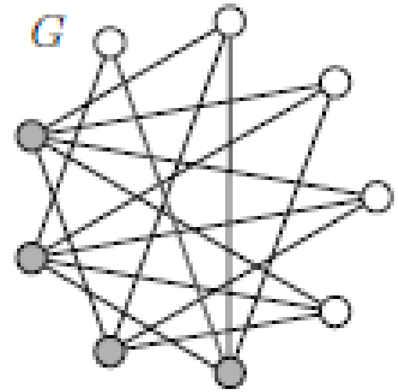
V' is a clique of size k in $\overline{G}$

$\Leftrightarrow$



V' is a independent set of size k in G

$\Leftrightarrow$



$V \setminus V'$ is a vertex cover of size $n - k$ in G

Clique, Independent Set, and Vertex Cover

Lemma: Given an undirected graph $G = (V, E)$ with n vertices and a subset $V' \subseteq V$ of size k. The following are equivalent:

i. V ′ is a clique of size k for the complement, G

ii. V ′ is an independent set of size k for G

iii. V \ V ′ is a vertex cover of size n − k for G, (where n = |V |)

Proof:

(i) ⇒ (ii): If V′ is a clique for G, then for each u, v ∈ V′, {u, v} is an edge of G implying that {u, v} is not an edge of G, implying that V′ is an independent set for G.

(ii) ⇒ (iii): If V′ is an independent set for G, then for each u, v ∈ V′, {u, v} is not an edge of G, implying that every edge in G is incident to a vertex in V \ V′, implying that V \ V′ is a vertex cover for G.

(iii) ⇒ (i): If V \ V′ is a vertex cover for G, then for any u, v ∈ V′ there is no edge {u, v} in G, implying that there is an edge {u, v} in G, implying that V′ is a clique in G.

CLIQUE is NP-Complete

Theorem: CLIQUE is NP-complete.

CLIQUE ∈ NP: We guess the k vertices that will form the clique. We can easily verify in polynomial time that all pairs of vertices in the set are adjacent (e.g., by inspection of $O(k^2)$ entries of the adjacency matrix).

IS $\leq_P$ CLIQUE: We want to show that given an instance of the IS problem (G, k), we can produce an equivalent instance of the CLIQUE problem in polynomial time. The reduction function f inputs G and k, and outputs the pair $(\bar{G}, k)$. Clearly this can be done in polynomial time. By the above lemma, this instance is equivalent.

VC is NP-complete

Theorem: VC is NP-complete.

VC ∈ NP: The certificate consists of the k vertices in the vertex cover. Given such a certificate we can easily verify in polynomial time that every edge is incident to one of these vertices.

IS $\leq_P$ VC: We want to show that given an instance of the IS problem (G, k), we can produce an equivalent instance of the VC problem in polynomial time. The reduction function f inputs G and k, computes the number of vertices, n, and then outputs (G, n − k). Clearly this can be done in polynomial time.