COMP 355
Advanced Algorithms

Dominating Set
Chapter 8 (KT)
Section 34.5(CLRS)

Rhodes College
—1848—

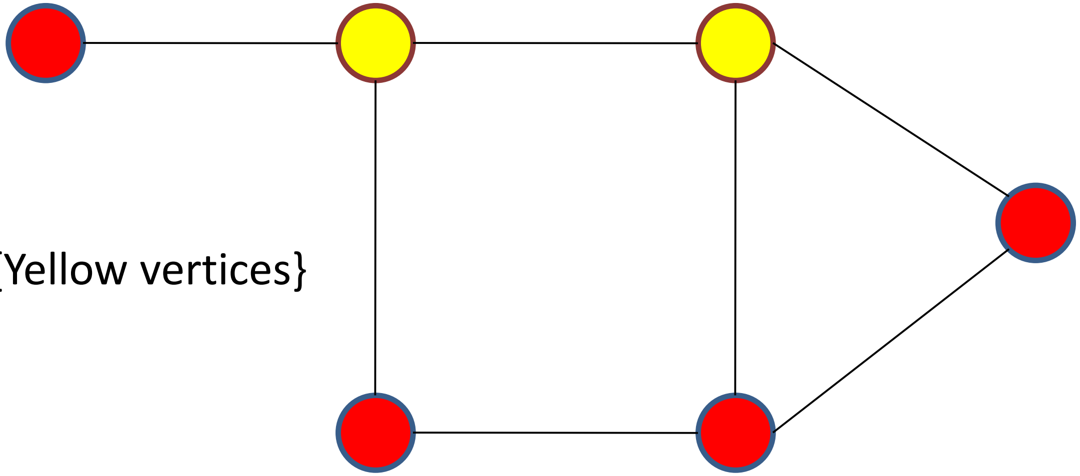
Dominating Set

Dominating-set = Given <G, k>, does a dominating set of size (at most) *k* for *G* exists?

Let *G* = (*V*, *E*) be an undirected graph

A dominating set *D* is a set of vertices that covers all vertices

- i.e., every vertex of *G* is either in *D* or is adjacent to at least one vertex from *D*

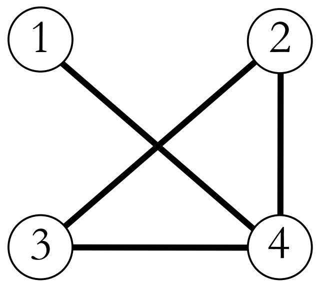


Size-2 example : {Yellow vertices}

[Recap] Vertex cover

A vertex cover, V’, is a set of vertices that covers all edges

- i.e., each edge is at least adjacent to some node in V’



{2, 4}, {3, 4}, {1, 2, 3}
are vertex covers

Decision Problem: Given an undirected graph G and an integer k, does G have a vertex cover of size k?

Dominating Set (Proof Sketch)

Steps:

1) Show that Dominating-set $\in$ NP.
2) Show that Dominating-set is not easier than a NPC problem
   - We choose this NPC problem to be Vertex cover
   - Reduction from Vertex-cover to Dominating-set
3) Show the correspondence of "yes" instances between the reduction

Dominating Set

Dominating Set: As with vertex cover, dominating set is an example of a graph covering problem.

- Each vertex is adjacent to at least one member of the dominating set, as opposed to each edge being incident to at least one member of the vertex cover.
- Obviously, if G is connected and has a vertex cover of size k, then it has a dominating set of size k (the same set of vertices), but the converse is not necessarily true.
- However, the similarity suggests that if VC is NP-complete, then DS is likely to be NP-complete as well.

Theorem: DS is NP-complete.

1. DS $\in$ NP.
2. VC $\leq_P$ DS

Dominating Set - (1) NP

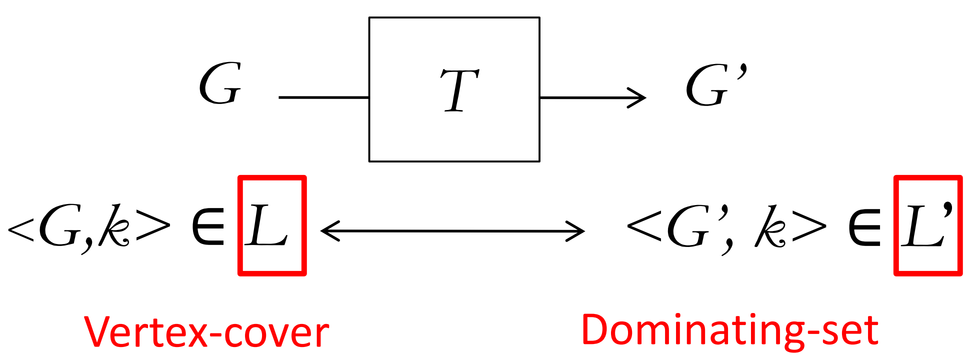
It is trivial to see that Dominating-set ∈ NP

- Given a vertex set D of size *k*, we check whether (V-D) are adjacent to D
- i.e., for each vertex, v, not in D, whether v is adjacent to some vertex u in D

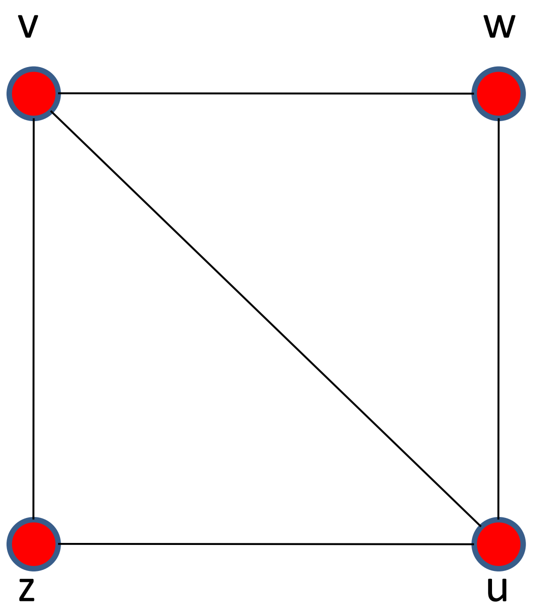
Dominating Set - (2) Reduction

Reduction - Graph transformation

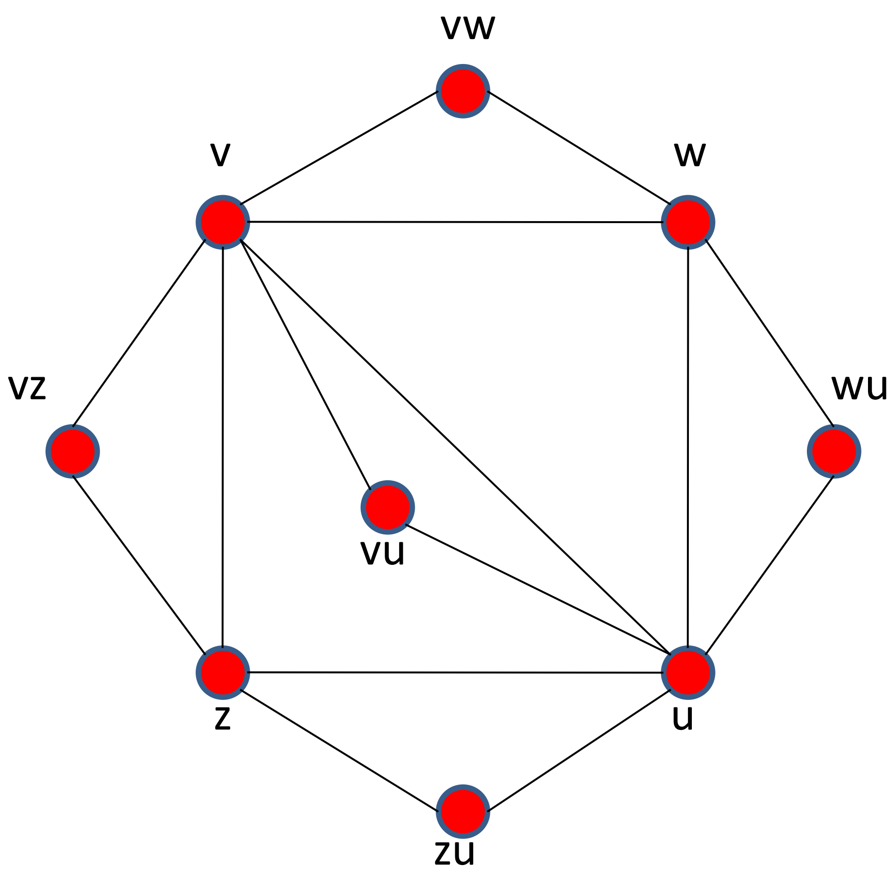
- For each edge (v, w) of G, add a vertex vw and the edges (v, vw) and (w, vw) to G'
- Furthermore, remove all vertices with no incident edges; such vertices would always have to go in a dominating set but are not needed in a vertex cover of G
  - We skip the discussion of this subtle part in the following



Dominating Set: Graph Transformation Example



G



G'

Dominating Set - (3) Correspondence

A dominating set of size *k* in *G'* ⇔ A vertex cover of size *K* in *G*

➔ Let D be a dominating set of size k in G'

- Case 1): D contains only vertices from G

  Then, all new vertices have an edge to a vertex in D

  D covers all edges

  D is a valid vertex cover of G

Dominating Set - (3) Correspondence

A dominating set of size *K* in *G'* ⇔ A vertex cover of size *K* in *G*

➔ Let D be a dominating set of size K in G'

- Case 2): D contains some new vertices (vertex in the form of uv)

  (We show how to construct a vertex cover using only old vertices, otherwise we cannot obtain a vertex cover for G)

  For each new vertex uv, replace it by u (or v)

  If u ∈ D, this node is not needed

  Then the edge u-v in G will be covered

  After new edges are removed, it is a valid vertex cover of G (of size at most K)

Dominating Set - (3) Correspondence

A dominating set of size *k* in *G'* ⇔ A vertex cover of size *k* in *G*

🡄 Let C be a vertex cover of size *k* in G

For an old vertex, v ∈ G' :

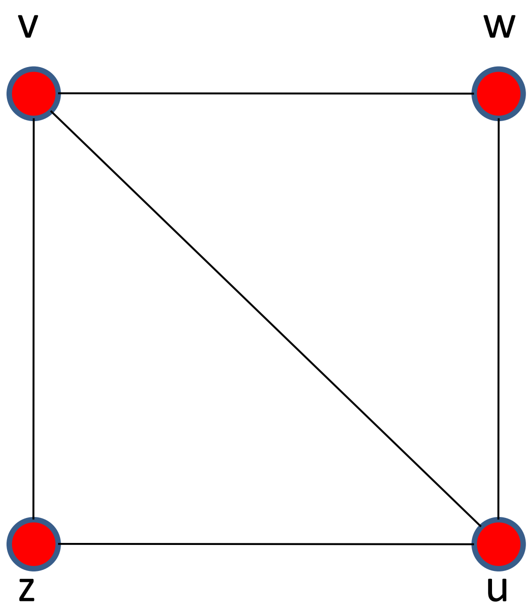
- By the definition of VC, all edges incident to v are covered
- v is also covered

For a new vertex, uv ∈ G' :

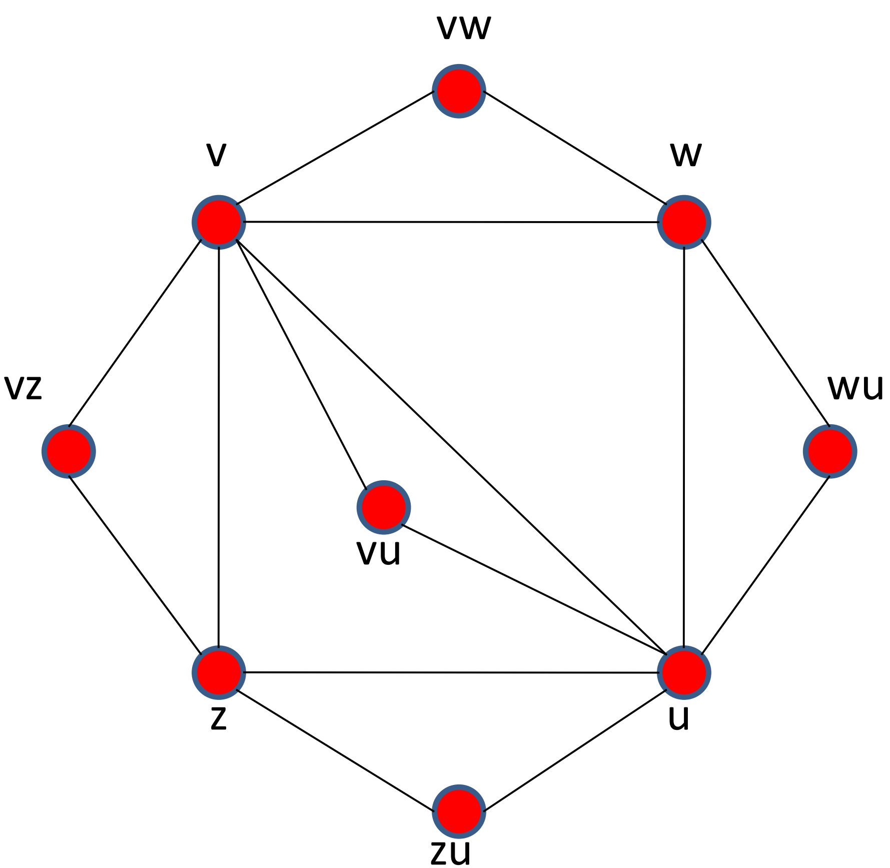
- Edge u-v must be covered, either u or v ∈ C
- This node will cover uv in G'

Thus, C is a valid dominating for G' (of size at most *k*)

Dominating Set: Graph Transformation Example

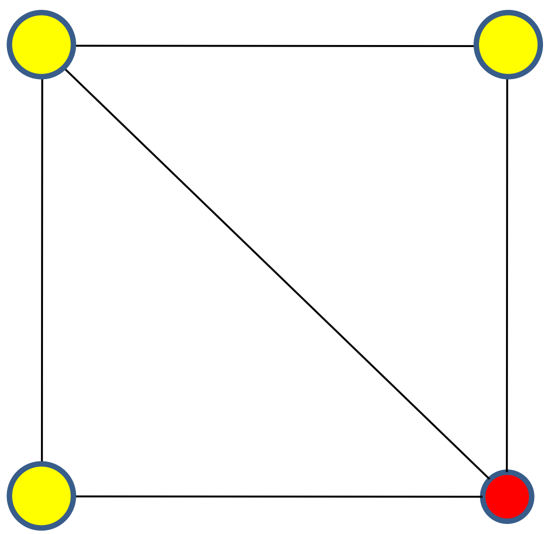


G

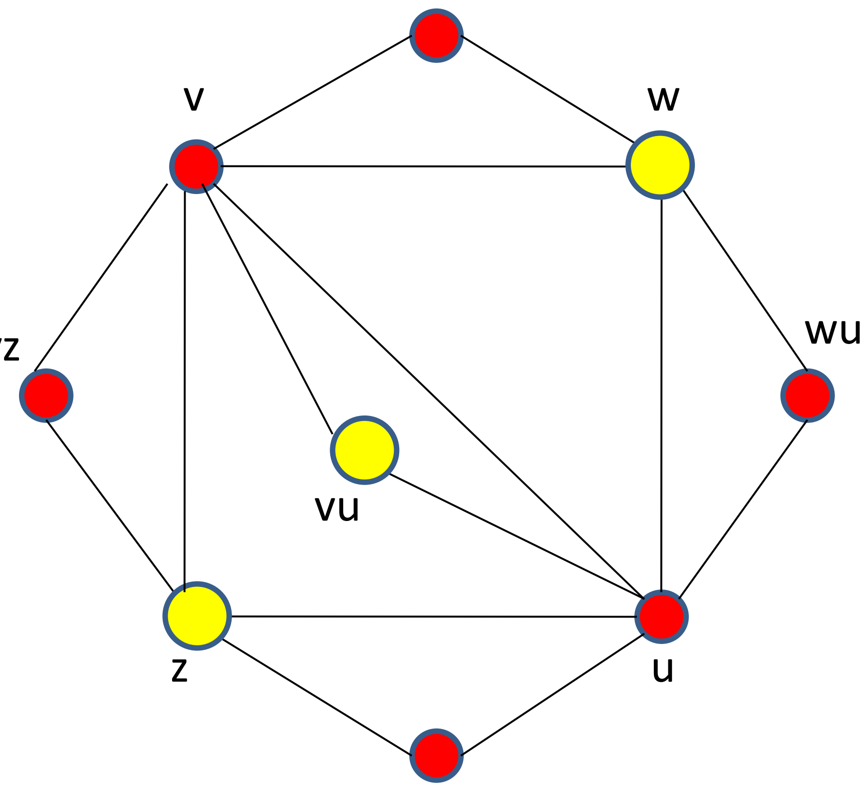


G'

Dominating Set - (3) Correspondence



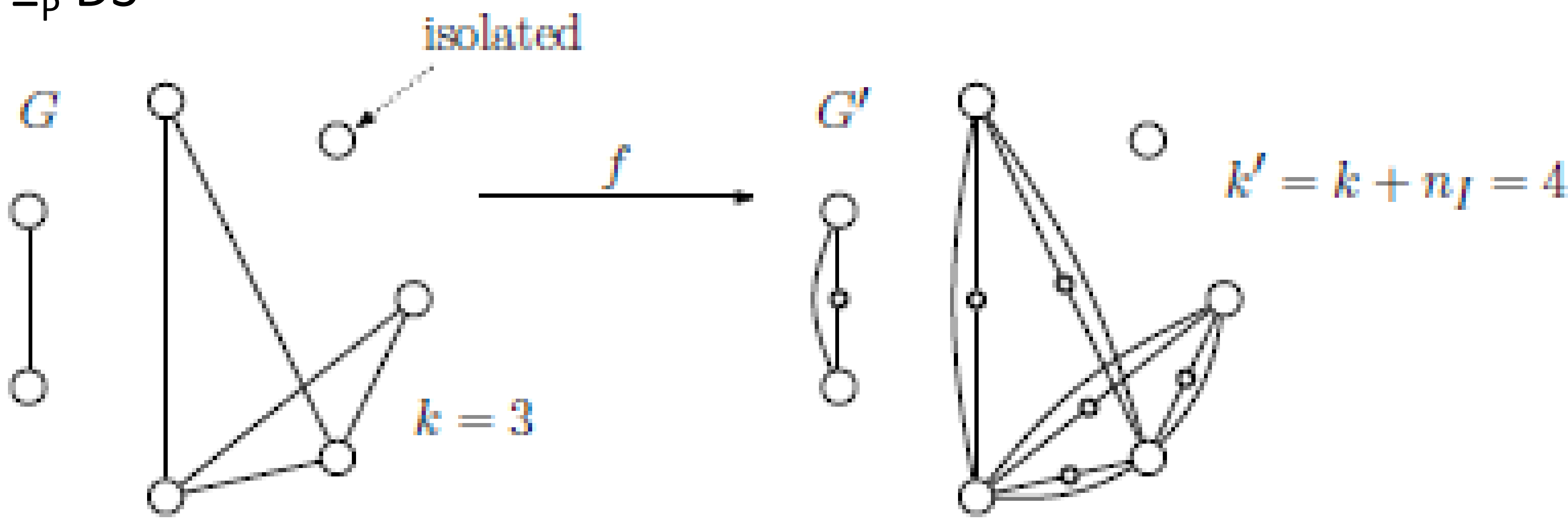
Vertex-cover *in G*



Dominating-set in *G′*

Vertex Cover to Dominating Set

VC $\leq_P$ DS



Dominating set reduction with k = 3 and one isolated vertex.

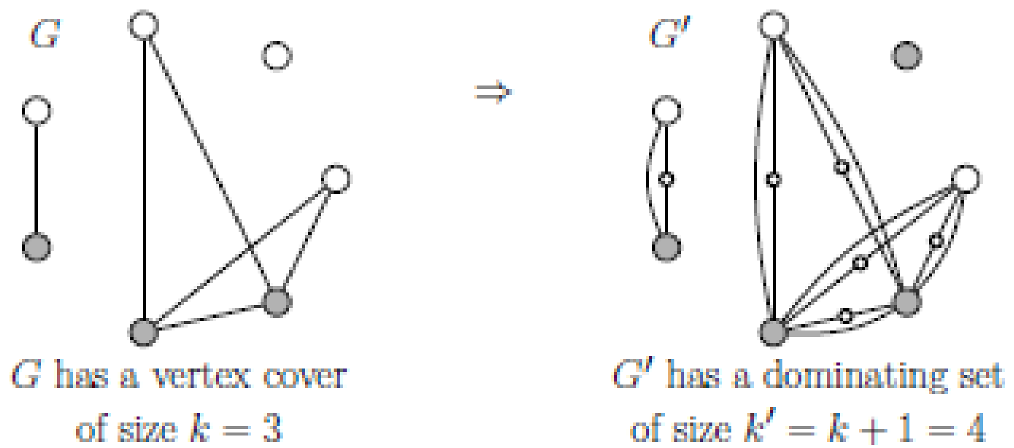
VC: “every edge is incident to a vertex in V′ ”.

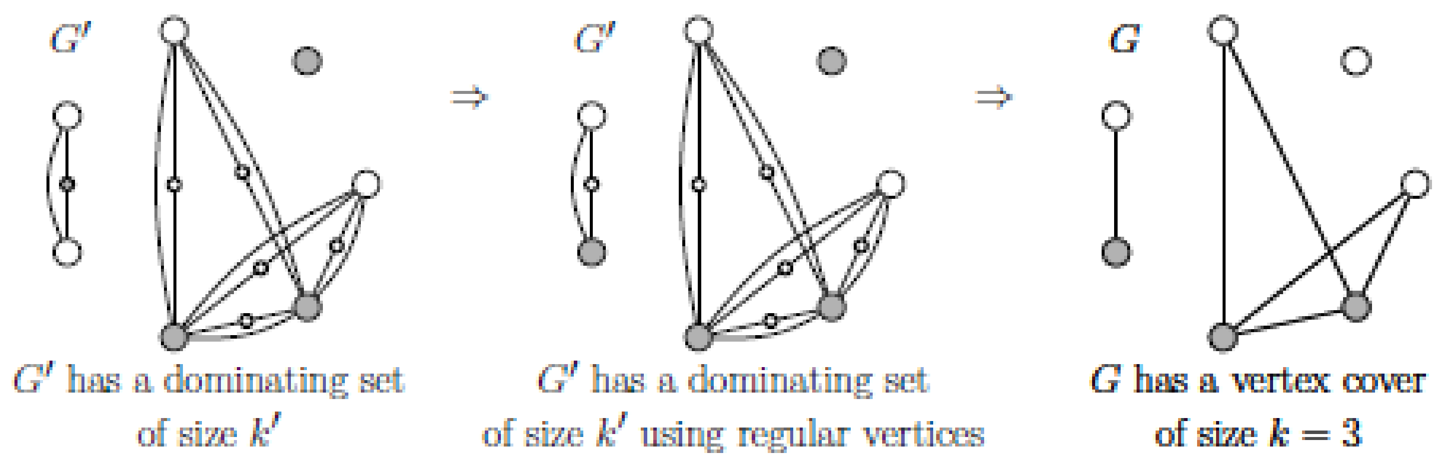
DS: “every vertex is either in V′ or is adjacent to a vertex in V′ ”.

Translation must somehow map the notion of “incident” to “adjacent”

Correctness of the Reduction

We need to show that G has a vertex cover of size k if and only if G′ has a dominating set of size k′.





Correctness of the VC to DS reduction (where k = 3 and l = 1).

NP-Completeness

So far, we have seen:

1. 3-SAT to INDEPENDENT SET (IS)
2. IS to CLIQUE
3. IS to VERTEX COVER
4. VERTEX COVER to DOMINATING SET
5. 3-COLORING to CLIQUE COVER (not the same as CLIQUE)

Practice

(KT Ch. 8, Problem 3)

Suppose you're helping to organize a summer sports camp, and the following problem comes up. The camp is supposed to have at least one counselor who's skilled at each of the *n* sports covered by the camp (baseball, volleyball, etc.). They have received job applications from *m* potential counselors. For each of the *n* sports, there is some subset of the m applicants qualified in that sport. The question is: For a given number $k < m$, is is possible to hire at most *k* of the counselors and have at least one counselor qualified in each of the *n* sports? We'll call this the *Efficient Recruiting Problem.*

Show that the *Efficient Recruiting Problem* is NP-Complete.

So your problem is NP-Complete? Now What?

Important: NP-Completeness is not a death sentence, but you need appropriate expectations/strategies

Some Useful Strategies

1. Brute-Force (for small input sizes)
2. Heuristics – Fast algorithms that are not always correct
3. Solve in exponential time but faster than brute-force search
4. Approximation Algorithms