COMP 355
Advanced Algorithms

Approximation Algorithms: VC and TSP

Chapter 11 (KT)

Section 35.1-35.2(CLRS)

Rhodes College
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Coping with NP-Completeness

Brute-force search: Viable option for small input sizes (e.g., $n \leq 20$).

Heuristics: Produces a valid solution, but no guarantee on how close it is to optimal.

General Search Algorithms: Examples: branch-and-bound, Metropolis-Hastings, simulated annealing, and genetic algorithms. Performance varies considerably from one problem to problem and instance to instance. But in some cases they can perform quite well.

Approximation Algorithms: Algorithm that runs in polynomial time (ideally), and produces a solution that is guaranteed to be within some factor of the optimum solution.

Approximation Algorithms

Q. Suppose I need to solve an NP-hard problem. What should I do?
A. Theory says you're unlikely to find a poly-time algorithm.

Must sacrifice one of three desired features.

- Solve problem to optimality.
- Solve problem in poly-time.
- Solve arbitrary instances of the problem.

ρ-approximation algorithm.

- Guaranteed to run in poly-time.
- Guaranteed to solve arbitrary instance of the problem
- Guaranteed to find solution within ratio ρ of true optimum.

Challenge. Need to prove a solution's value is close to optimum, without even knowing what optimum value is!

Performance Bounds

How do we measure how good an approximation algorithm is?

- Given an instance I of our problem, let C(I) be the cost of the solution produced by our approximation algorithm, and let C∗(I) be the optimal solution. (assume that costs are strictly positive values.)
  - For a minimization problem we have C(I)/C∗(I) ≥ 1.
  - For a maximization problem we have C∗(I)/C(I) ≥ 1.
  - In either case, we want the ratio to be as small as possible.
- For any input size n, we say that the approximation algorithm achieves performance ratio bound ρ(n), if for all I, |I| = n we have:

$$\max_I \left(\frac{C(I)}{C^*(I)}, \frac{C^*(I)}{C(I)} \right) \; \leq \; \rho(n).$$

Performance Bounds

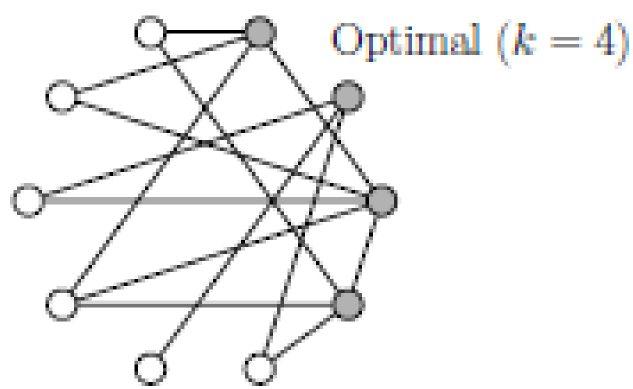
Some NP-complete are *inapproximable*: no polynomial time algorithm achieves a ratio bound smaller than ∞ unless P = NP.

Some NP-Complete can be approximated:

- Ratio bound is a function of *n*.
- Ratio bound is a constant.
- Approximated arbitrarily well. In particular, the user provides a parameter $\varepsilon > 0$ and the algorithm achieves a ratio bound of $(1+\varepsilon)$. Of course, as ε approaches 0 the algorithm's running time gets worse. If such an algorithm runs in polynomial time for any fixed ε, it is called a polynomial time approximation scheme (PTAS).

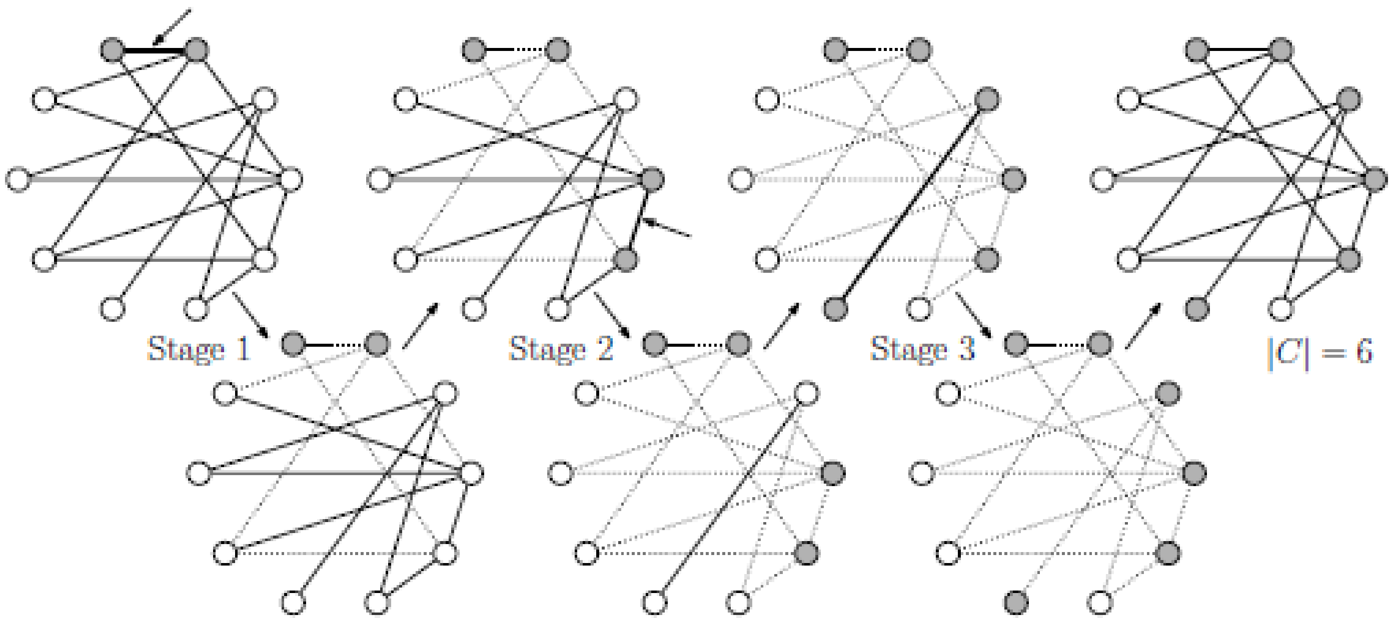
Vertex Cover

- There is an algorithm for vertex cover with a ratio bound of 2.
- This algorithm will be guaranteed to find a vertex cover whose size is at most twice that of the optimum.
- Recall that a vertex cover is a subset of vertices such that every edge in the graph is incident to at least one of these vertices.
- The vertex cover optimization problem is to find a vertex cover of minimum size



Vertex cover (optimal solution)

The 2-for-1 heuristic for VC



2-for-1 Approximation for VC

```
   two-for-one-VC(G=(V,E)) {
       C = empty
       while (E is nonempty) do {
(*)        let (u,v) be any edge of E
           add both u and v to C
           remove from E all edges incident to either u or v
       }
       return C
   }
```

2-for-1 Approximation for VC

Claim: The 2-for-1 approximation for VC achieves a performance ratio of 2.

Proof: returns a vertex cover for G that is at most twice the size of the optimal vertex cover. Consider the set C output by two-for-one-VC(G). Let C^* be the optimum vertex cover. Let A be the set of edges selected by the line marked with "$(*)$" in the code fragment. Because we add both endpoints of each edge of A to C, we have $|C| = 2|A|$. However, the optimum vertex cover C^* must contain at least one of these two vertices. Therefore, we have $|C^*| \geq |A|$. Therefore

$$|C| \; = \; 2|A| \; \leq \; 2|C^*| \qquad \Rightarrow \qquad \frac{|C|}{|C^*|} \; \leq \; 2$$

as desired.

Reductions and Approximations

Approximation factors are not generally preserved by transformations.

Example: Recall that if V' is a vertex cover for G, then the complement vertex set, $V \setminus V'$, is an independent set for G.

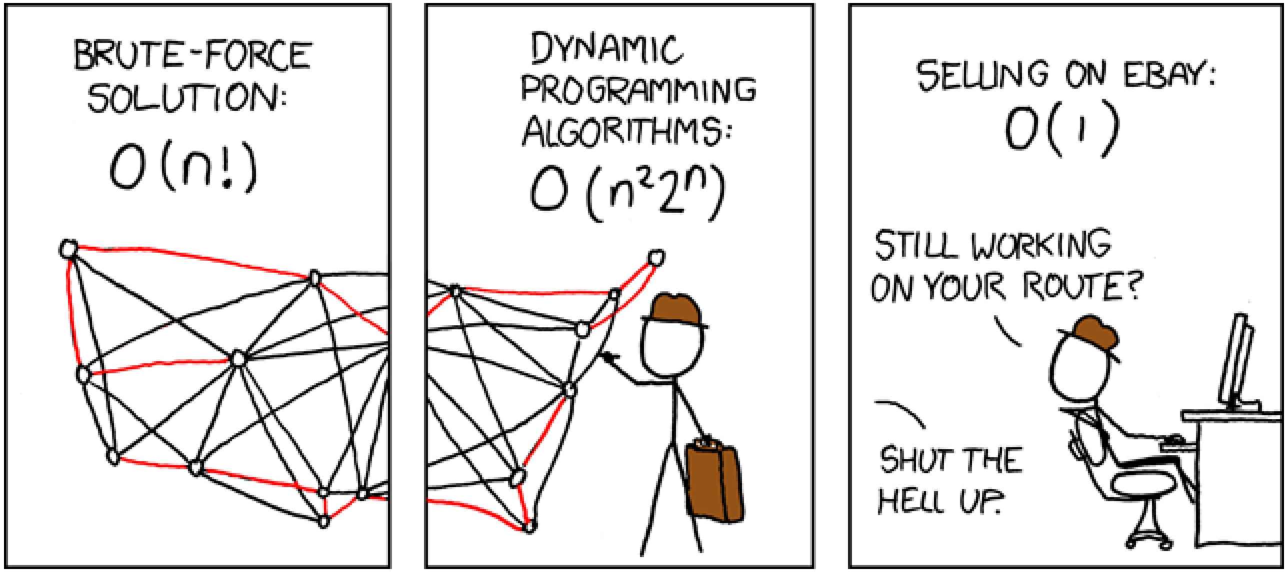
- Suppose that G has n vertices, and a minimum vertex cover V' of size k. Then our heuristic is guaranteed to produce a vertex cover V'' that is of size at most $2k$.
- If we consider the complement set $V \setminus V'$, we know that G has a maximum independent set of size $n - k$.
- By complementing our approximation V \ V'' we have an "approximate" independent set of size $n - 2k$.
- How good is this?
- Performance ratio: $\rho(n, k) = \dfrac{n-k}{n-2k}$

The problem is that this ratio may be arbitrarily large. For example, if $n = 1001$ and $k = 500$, then the ratio is $501/(1001 - 1000) \approx 500/1 = 500$.

Traveling Salesman Problem

Traveling Salesperson Decision Problem (TSP) - Given a **complete** undirected graph with nonnegative edge weights, does there exist a cycle that visits all vertices and costs <= k?

- Let w(u, v) denote the weight on edge (u, v).
- Given a set of edges A forming a tour we define W(A) to be the sum of edge weights in A.



Traveling Salesman with Triangle Inequality

Traveling Salesperson Optimization Problem (TSP) - Given a complete undirected graph with nonnegative edge weights, and find a cycle that visits all vertices and is of minimum cost.

- Let w(u, v) denote the weight on edge (u, v).
- Given a set of edges A forming a tour we define W(A) to be the sum of edge weights in A.

Many of the applications of TSP, the edge weights satisfy a property called the triangle inequality

$$\text{for all } u, v, x \in V,\ w(u, v) \leq w(u, x) + w(x, v).$$

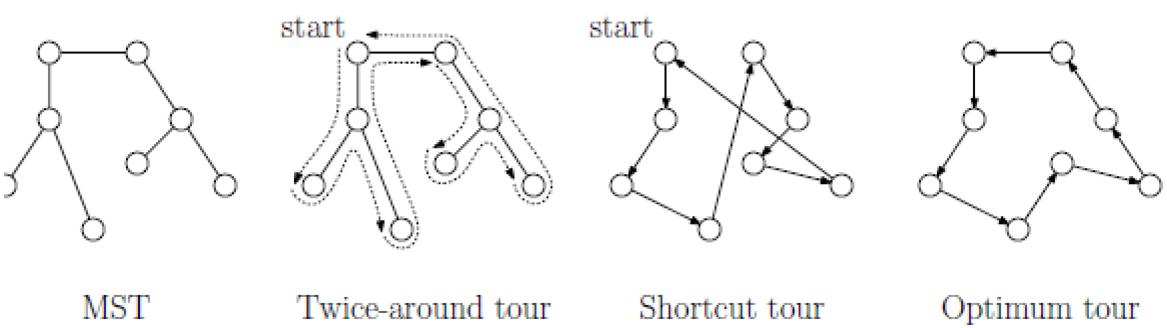
When the underlying cost function satisfies the triangle inequality there is an approximation algorithm for TSP with a ratio-bound of 2.

- The key insight is to observe that a TSP with one edge removed is just a spanning tree (not necessarily a MST).
- Cost of the minimum TSP tour is at least as large as the cost of the MST
- If we can find some way to convert the MST into a TSP tour while increasing its cost by at most a constant factor, then we will have an approximation for TSP
- If edge weights satisfy triangle inequality, this is possible.

Traveling Salesman with Triangle Inequality

TSP Approximation

```
approx-TSP(G=(V,E)) {
    T = minimum spanning tree for G
    r = any vertex
    H = list of vertices visited by a preorder walk of T starting at r
    return L
}
```



TSP Approximation

Approx-TSP Performance Ratio

Claim: Approx-TSP achieves a performance ratio of 2.

Proof: Let H denote the tour produced by this algorithm and let H^* be the optimum tour. Let T be the minimum spanning tree. As we said before, since we can remove any edge of H^* resulting in a spanning tree, and since T is the minimum cost spanning tree we have

$$W(T) \leq W(H^*).$$

Now observe that the twice around tour of T has cost $2 \cdot W(T)$, since every edge in T is hit twice. By the triangle inequality, when we short-cut an edge of T to form H we do not increase the cost of the tour, and so we have

$$W(H) \leq 2 \cdot W(T).$$

Combining these we have

$$W(H) \leq 2 \cdot W(T) \leq 2 \cdot W(H^*) \Rightarrow \frac{W(H)}{W(H^*)} \leq 2,$$

as desired.

Practice

1. Give an example of a graph for which the 2-for-1 VC algorithm yields a suboptimal solution.

2. We know that both the VERTEX COVER problem and the CLIQUE problem are NP-Complete, and as we showed previously, they are complementary in the sense that a minimum-size vertex cover is the complement of a maximum-size clique in the complementary graph.

Given the 2-for-1 VC algorithm, does the above relationship imply that there is a polynomial-time approximation algorithm with a constant approximation-ratio for the CLIQUE problem?