Problem Set 10: More NP-Completeness

Handed out Friday, November 8. Due at the start of class Friday, November 15.

Problem 1.(10 points) Given a directed graph $G = (V, E)$, a *vertex cycle cover* is a subset of vertices such that every simple cycle in G passes through at least one of these vertices. (For example, the graph shown in Fig 1 has a vertex cycle cover of size 2 (shaded).)

Vertex Cycle Cover (VCC): Given a digraph G and an integer k, does G contain a vertex cycle cover of size at most k?

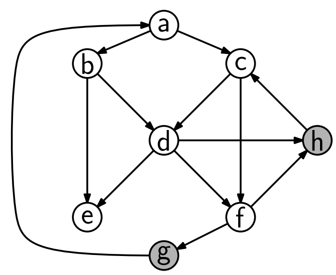


Figure 1: Problem 1: A digraph and a vertex cycle cover consisting of $\{g, h\}$.

Show that VCC is in NP. (You do *not* need to show that it is NP-hard, but see the challenge problem.)

Hint: Be careful. A graph can have exponentially many simple cycles, so a naive implementation will take exponential time.

Problem 2.(15 points) When finding cliques, it is natural to look for vertices of high degree. Suppose, however, that you want to find cliques consisting of vertices of relatively low degree. We will show that even this problem is NP-complete.

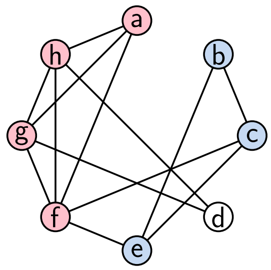
Low-Degree Clique (LDC): Given a graph $G = (V, E)$ and an integer k, does G have a clique of size at least k consisting entirely of vertices who degree is not greater than the median vertex degree of the entire graph?

By the *median vertex degree*, we mean the median value of the degrees of all n vertices of the graph. (For our purposes, we define the *median* of a set of n numbers to be the $\lceil n/2 \rceil$-smallest value of the set.)

For example, the graph shown in Fig. 2 has median vertex degree of 3. There exists an LDC of size 3 (vertices $\{b, c, e\}$) since all these vertices have degree at most 3. Even though there is a clique of size 4 (vertices $\{a, f, g, h\}$) it is not an LDC since it contains (at least one) vertex of degree higher than 3.

Show that the LDC problem is NP-Complete. Remember that this involves two things. Showing that LDC is in NP, and that some known NP-complete problem is polynomial time reducible to LDC.

Hint: Reduction from the standard Clique problem.



vertex:	a	b	c	d	e	f	g	h
degree:	3	2	3	2	3	5	4	4
median degree:	3							

Figure 2: Problem 2: Low-Degree Clique.

Challenge Problem

(Remember that challenge problems count for extra credit points.)

In reference to problem (1), show that the VCC problem is NP-complete, by showing that some known NP-complete problem can be reduced to it. (**Hint:** Reduction from either Vertex Cover or Independent Set.)