Problem Set 9: NP-Completeness

Handed out Friday, November 1. Due at the start of class Friday, November 8.

Problem 1. (15 points) Recall that a boolean formula is said to be in *conjunctive normal form* (CNF), if it is the logical-and of a set of clauses, where each clause is the logical-or of a set of literals, and each literal is a variable or its negation. Suppose that you are given a boolean formula F in CNF (where each clause may one, two, three, or more literals). An example of such a formula is shown below.

$$F = (a \vee b \vee \overline{c} \vee \overline{d}) \wedge (e \vee \overline{b}) \wedge (d) \wedge (\overline{a} \vee e \vee f \vee \overline{g} \vee \overline{c}).$$

(a) Explain how to convert any such formula F into an *equivalent* 3-CNF formula F'. This means that F' has *exactly* three literals per clause, and F' is satisfiable if and only if F is satisfiable. Your conversion should run in polynomial time. Briefly justify the correctness of your construction.

Hint: A 3-CNF clause is allowed to contain multiple copies of the same literal. Your conversion process is allowed to create new variables.

(b) Show the result of your transformation on the specific formula F above.

(c) Can you further enhance your transformation to convert any CNF formula in polynomial time to an equivalent one in 2-CNF (exactly two literals per clause)? If possible, explain how to do this. If not, explain what goes wrong when you try to adapt your solution to part (a).

Problem 2. (20 points) Highly intelligent aliens land on Earth and tell us the following two things and then leave immediately.

(a) The 3-Coloring problem (which is NP-complete) is solvable in worst-case $O(n^9)$ time, where n denotes the number of vertices in the graph.

(b) There is no algorithm for 3-Coloring that runs faster than $\Omega(n^7)$ time in the worst case.

Assuming these two facts, for each of the following assertions, indicate whether it can be inferred from the information the aliens have given us. (In all cases, time complexities are understood to be *worst-case* running time.) Provide a short justification in each case.

(i) All *NP-complete* problems are solvable in polynomial time.

(ii) All problems in NP, even those that are *not* NP-complete, are solvable in polynomial time.

(iii) All *NP-hard* problems are solvable in polynomial time.

(iv) All NP-complete problems are solvable in $O(n^9)$ time.

(v) No NP-complete problem can be solved faster than $\Omega(n^7)$ time.