COMP 355
Advanced Algorithms

Greedy Algorithms for Scheduling
Sections 16.1-16.2

Rhodes College
Greedy Algorithms

• Optimization problem
  – Given an input, either minimize or maximize profit.

• Efficiency
  – Can we produce the optimal solution without using brute-force?

• Work for a number of optimization problems including MSTs (optimal solution)

• Provide fast heuristics (non-optimal solution strategies) = good approximations
Goal. Given currency denominations: 1, 5, 10, 25, 100, devise a method to pay amount to customer using fewest number of coins.

Ex: 34¢.

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Ex: $2.89
Coin-Changing: Greedy Algorithm

Cashier's algorithm. At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Sort coins denominations by value: $c_1 < c_2 < \ldots < c_n$.

coins selected

S $\leftarrow$ $\emptyset$

while (x $\neq$ 0) {
    let k be largest integer such that $c_k \leq x$
    if (k = 0)
        return "no solution found"
    x $\leftarrow$ x $-$ $c_k$
    S $\leftarrow$ S $\cup$ {k}
} return S

Q. Is cashier's algorithm optimal?
Coin-Changing: Analysis of Greedy Algorithm

**Theorem.** Greed is optimal for U.S. coinage: 1, 5, 10, 25, 100.

**Pf.** (by induction on x)
- Consider optimal way to change \( c_k \leq x < c_{k+1} \): greedy takes coin \( k \).
- We claim that any optimal solution must also take coin \( k \).
  - if not, it needs enough coins of type \( c_1, \ldots, c_{k-1} \) to add up to \( x \)
  - table below indicates no optimal solution can do this
- Problem reduces to coin-changing \( x - c_k \) cents, which, by induction, is optimally solved by greedy algorithm.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( c_k )</th>
<th>All optimal solutions must satisfy</th>
<th>Max value of coins 1, 2, ..., k-1 in any OPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( P \leq 4 )</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>( N \leq 1 )</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>( N + D \leq 2 )</td>
<td>( 4 + 5 = 9 )</td>
</tr>
<tr>
<td>4</td>
<td>25</td>
<td>( Q \leq 3 )</td>
<td>( 20 + 4 = 24 )</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>no limit</td>
<td>( 75 + 24 = 99 )</td>
</tr>
</tbody>
</table>
Observation. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.
- Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
- Optimal: 70, 70.
Interval Scheduling

Interval scheduling.
• Job $j$ starts at $s_j$ and finishes at $f_j$.
• Two jobs **compatible** if they don't overlap.
• **Goal**: find maximum subset of mutually compatible jobs.
Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- **[Earliest start time]** Consider jobs in ascending order of start time $s_j$.

- **[Shortest interval]** Consider jobs in ascending order of interval length $f_j - s_j$.

- **[Fewest conflicts]** For each job, count the number of conflicting jobs $c_j$. Schedule in ascending order of conflicts $c_j$. 

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- Breaks earliest start time
- Breaks shortest interval
- Breaks fewest conflicts
Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

Sort jobs by finish times so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

```
A ← φ
for j = 1 to n {
    if (job j compatible with A)
        A ← A ∪ {j}
}
return A
```

Implementation. $O(n \log n)$.

- Remember job $j^*$ that was added last to $A$.
- Job $j$ is compatible with $A$ if $s_j \geq f_{j^*}$. 
Interval Scheduling
Interval Scheduling
Interval Scheduling

Time

0 1 2 3 4 5 6 7 8 9 10 11

A B C D E F G H

0 1 2 3 4 5 6 7 8 9 10 11

A B
Interval Scheduling

A, B, C, D, E, F, G, H

Time
Interval Scheduling

Time

Interval Scheduling

1/10/2014
Interval Scheduling

Time

A

B

C

D

E

F

G

H

0 1 2 3 4 5 6 7 8 9 10 11

1/10/2014
Interval Scheduling
Interval Scheduling
Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

– Assume greedy is not optimal, and let's see what happens.
– Let \( i_1, i_2, \ldots i_k \) denote set of jobs selected by greedy.
– Let \( j_1, j_2, \ldots j_m \) denote set of jobs in the optimal solution with \( i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r \) for the largest possible value of \( r \).
Interval Partitioning

Interval partitioning.

- Lecture $j$ starts at $s_j$ and finishes at $f_j$.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

**Ex:** This schedule uses 4 classrooms to schedule 10 lectures.
Interval Partitioning

Interval partitioning.

- Lecture j starts at $s_j$ and finishes at $f_j$.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.
**Def.** The *depth* of a set of open intervals is the maximum number that contain any given time.

**Key observation.** Number of classrooms needed $\geq$ depth.

**Ex:** Depth of schedule below $= 3 \implies$ schedule below is optimal.

$a, b, c$ all contain 9:30

**Q.** Does there always exist a schedule equal to depth of intervals?
Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

Sort intervals by starting time so that \( s_1 \leq s_2 \leq \ldots \leq s_n \).

\( d \leftarrow 0 \) \hspace{1cm} \text{number of allocated classrooms}

\begin{verbatim}
for j = 1 to n {
    if (lecture j is compatible with some classroom k)
        schedule lecture j in classroom k
    else
        allocate a new classroom \( d + 1 \)
        schedule lecture j in classroom \( d + 1 \)
        \( d \leftarrow d + 1 \)
}
\end{verbatim}

Implementation. \( O(n \log n) \).
- For each classroom \( k \), maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.
Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.

Pf.

- Let $d =$ number of classrooms that the greedy algorithm allocates.
- Classroom $d$ is opened because we needed to schedule a job, say $j$, that is incompatible with all $d-1$ other classrooms.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than $s_j$.
- Thus, we have $d$ lectures overlapping at time $s_j + \varepsilon$.
- Key observation $\Rightarrow$ all schedules use $\geq d$ classrooms.
Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires $t_j$ units of processing time and is due at time $d_j$.
- If j starts at time $s_j$, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$.
- Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

Ex:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_j$</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$d_j$</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

lateness = 2  lateness = 0  max lateness = 6

<table>
<thead>
<tr>
<th>$d_3 = 9$</th>
<th>$d_2 = 8$</th>
<th>$d_6 = 15$</th>
<th>$d_1 = 6$</th>
<th>$d_5 = 14$</th>
<th>$d_4 = 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time $t_j$.

- [Earliest deadline first] Consider jobs in ascending order of deadline $d_j$. 

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$. 
Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time $t_j$.

```
<table>
<thead>
<tr>
<th>t_j</th>
<th>d_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
</tbody>
</table>
```

Counterexample

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

```
<table>
<thead>
<tr>
<th>t_j</th>
<th>d_j</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>
```

Counterexample
Minimizing Lateness: Greedy Algorithms

**Greedy algorithm.** Earliest deadline first.

Sort $n$ jobs by deadline so that $d_1 \leq d_2 \leq \ldots \leq d_n$

$t \leftarrow 0$

for $j = 1$ to $n$

Assign job $j$ to interval $[t, t + t_j]$

$s_j \leftarrow t$, $f_j \leftarrow t + t_j$

$t \leftarrow t + t_j$

output intervals $[s_j, f_j]$

<table>
<thead>
<tr>
<th>$t_j$</th>
<th>3</th>
<th>2</th>
<th>1</th>
<th>4</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_j$</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

$\text{max lateness} = 1$

| $d_1 = 6$ | $d_2 = 8$ | $d_3 = 9$ | $d_4 = 9$ | $d_5 = 14$ | $d_6 = 15$ |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
Observation. There exists an optimal schedule with no idle time.

Observation. The greedy schedule has no idle time.
Minimizing Lateness: Inversions

**Def.** An *inversion* in schedule $S$ is a pair of jobs $i$ and $j$ such that: $i < j$ but $j$ scheduled before $i$.

**Observation.** Greedy schedule has no inversions.

**Observation.** If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.
**Def.** An inversion in schedule S is a pair of jobs i and j such that: i < j but j scheduled before i.

**Claim.** Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

**Pf.** Let \( \ell \) be the lateness before the swap, and let \( \ell' \) be it afterwards.

- \( \ell'_k = \ell_k \) for all \( k \neq i, j \)
- \( \ell'_i \leq \ell_i \)
- If job j is late:

\[
\ell'_j = f'_j - d_j = f_i - d_j \leq f_i - d_i \leq \ell_i
\] (definition) (\( j \) finishes at time \( f_i \)) (\( i < j \)) (definition)

1/10/2014
Minimizing Lateness: Analysis of Greedy Algorithm

**Theorem.** Greedy schedule $S$ is optimal.

**Pf.** Define $S^*$ to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can assume $S^*$ has no idle time.
- If $S^*$ has no inversions, then $S = S^*$.
- If $S^*$ has an inversion, let $i-j$ be an adjacent inversion.
  - swapping $i$ and $j$ does not increase the maximum lateness and strictly decreases the number of inversions
  - this contradicts definition of $S^*$
**Greedy Analysis Strategies**

*Greedy algorithm stays ahead.* Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

*Exchange argument.* Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

*Structural.* Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.
Next Time

• Greedy Algorithms: Huffman Coding
• Section 16.3