

COMP 355

Advanced Algorithms

Bellman-Ford Algorithm for Shortest Paths
Sections 6.8 (KT)



1

Shortest Path Applications

- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.



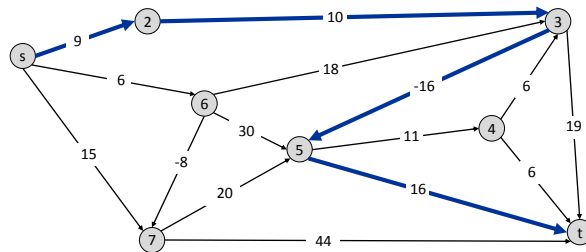
Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

2

Shortest Paths

Shortest path problem. Given a directed graph $G = (V, E)$, with edge weights c_{vw} find shortest path from node s to node t .
allow negative weights

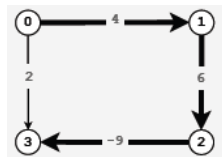
Ex. Nodes represent agents in a financial setting and c_{vw} is cost of transaction in which we buy from agent v and sell immediately to w .



3

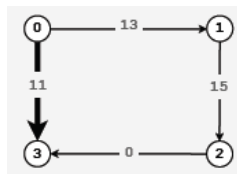
Shortest Paths: Failed Attempts

Dijkstra. Can fail if negative edge costs.



Dijkstra selects vertex 3 immediately after 0.
But shortest path from 0 to 3 is $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$.

Re-weighting. Adding a constant to every edge weight can fail.



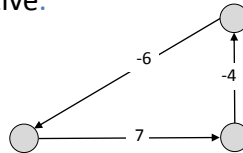
Adding 9 to each edge weight changes the shortest path from $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$ to $0 \rightarrow 3$.

Bad news: Need a different algorithm

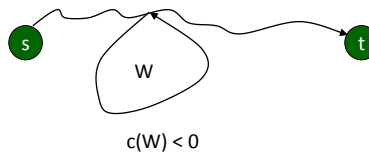
4

Shortest Paths: Negative Cost Cycles

Def. A negative cycle is a directed cycle such that the sum of its edge weights is negative.



Observation. If some path from s to t contains a negative cost cycle, there does not exist a shortest s - t path; otherwise, there exists one that is simple.



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Shortest Paths: Dynamic Programming

Def. $OPT(i, v)$ = length of shortest v - t path P using at most i edges.

- Case 1: P uses at most $i-1$ edges.
 - $OPT(i, v) = OPT(i-1, v)$
- Case 2: P uses exactly i edges.
 - if (v, w) is first edge, then OPT uses (v, w) , and then selects best w - t path using at most $i-1$ edges

$$OPT(i, v) = \begin{cases} 0 & \text{if } i = 0 \\ \min \left\{ OPT(i-1, v), \min_{(v, w) \in E} \{ OPT(i-1, w) + c_{vw} \} \right\} & \text{otherwise} \end{cases}$$

Remark. By previous observation, if no negative cycles, then $OPT(n-1, v)$ = length of shortest v - t path.

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Shortest Paths: Implementation

```

Shortest-Path( $G, s$ ) {
  foreach node  $v \in V$ 
     $M[0, v] \leftarrow \infty$ 
   $M[0, s] \leftarrow 0$ 

  for  $i = 1$  to  $n-1$ 
    foreach node  $v \in V$ 
       $M[i, v] \leftarrow M[i-1, v]$ 
      foreach edge  $(w, v) \in E$ 
         $M[i, v] \leftarrow \min \{ M[i, v], M[i-1, w] + c_{vw} \}$ 
}

```

- Analysis. $\Theta(mn)$ time, $\Theta(n^2)$ space.
- Finding the shortest paths. Maintain a "successor" for each table entry.

7

Bellman-Ford Algorithm

Bellman-Ford algorithm applies a relaxation to *every* edge in the graph, and repeats this $V - 1$ times.

Bellman-Ford Algorithm

```

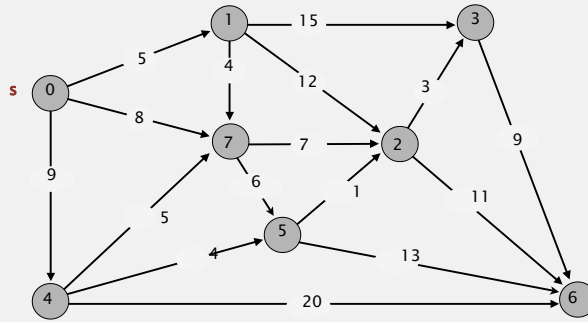
BellmanFord( $G, w, s$ ) {
  for each ( $u$  in  $V$ ) {
     $d[u] = +\text{infinity}$            // standard initialization
     $\text{pred}[u] = \text{null}$ 
  }
   $d[s] = 0$ 
  for  $i = 1$  to  $V-1$  {
    for each  $(u, v)$  in  $E$  {
      Relax( $u, v$ )              // repeat  $V-1$  times
                                // relax along each edge
    }
  }
}

```

The running time is $O(n*m)$, where $n = |V|$ and $m = |E|$, since there are two main nested loops, one iterated $V - 1$ times and the other iterated E times.

8

Bellman-Ford algorithm

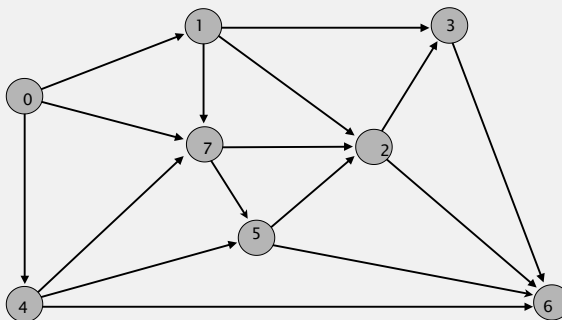
Repeat V times: relax all E edges.

an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

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Bellman-Ford algorithm

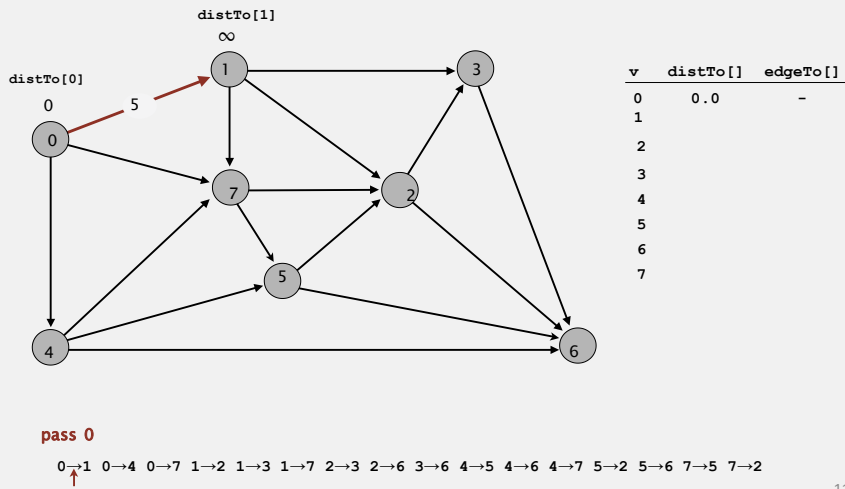
Repeat V times: relax all E edges.

Initialize

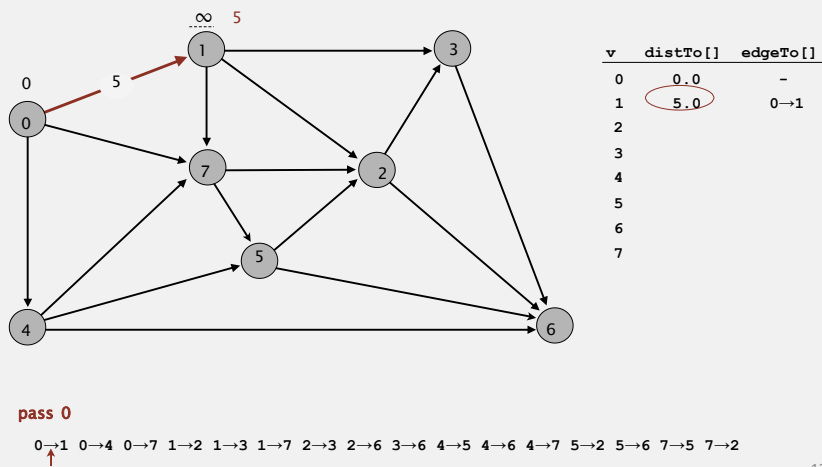
v	distTo[]	edgeTo[]
0	0.0	-
1		
2		
3		
4		
5		
6		
7		

10

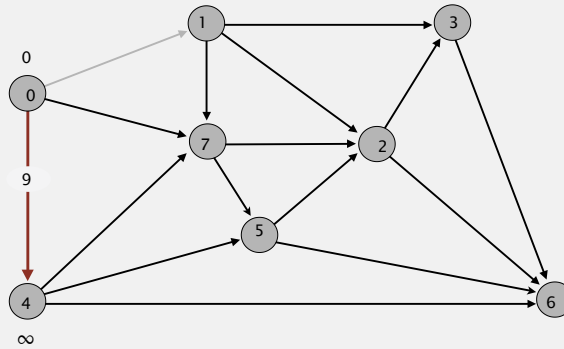
Bellman-Ford algorithm

Repeat V times: relax all E edges.

Bellman-Ford algorithm

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Repeat V times: relax all E edges.

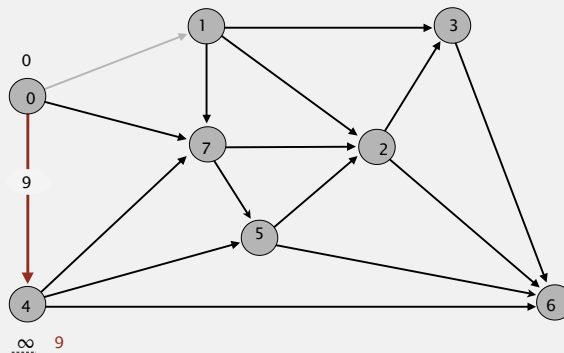
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4		
5		
6		
7		

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

13

Bellman-Ford algorithm

Repeat V times: relax all E edges.

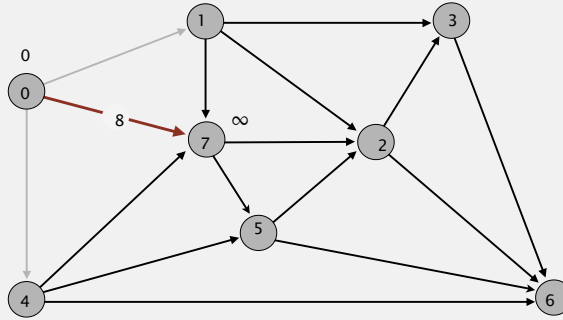
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7		

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

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Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7		

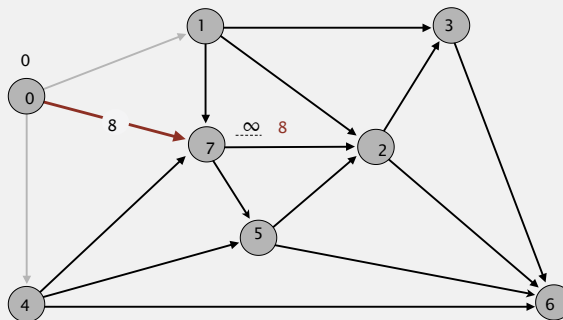
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

15

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

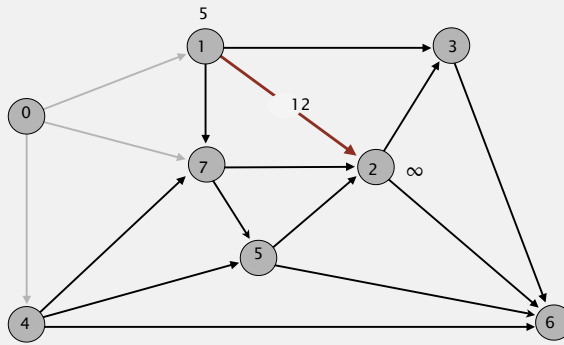
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

16

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

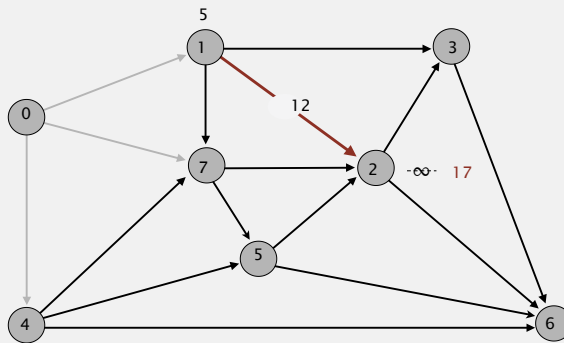
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

17

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

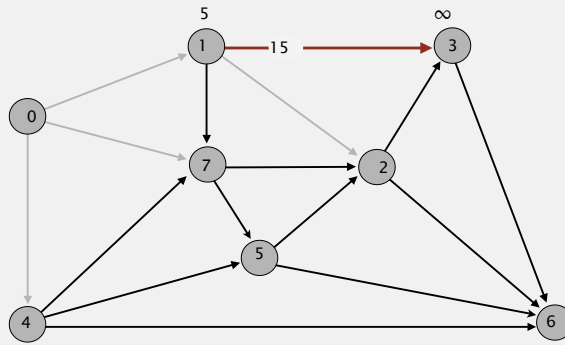
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

18

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	∞	
4	9.0	0→4
5	∞	
6	∞	
7	8.0	0→7

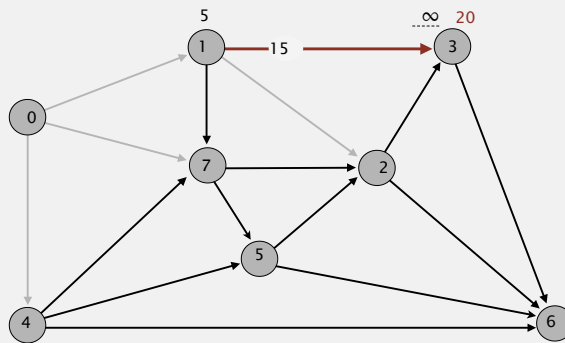
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

19

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	∞	
6	∞	
7	8.0	0→7

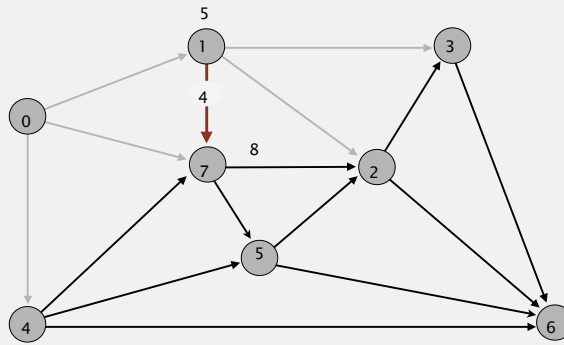
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

20

Bellman-Ford algorithm

Repeat V times: relax all E edges.

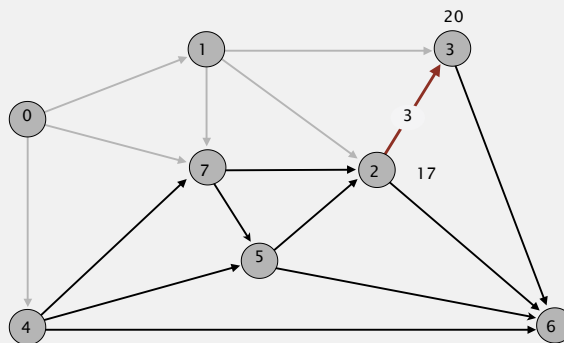
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

21

Bellman-Ford algorithm

Repeat V times: relax all E edges.

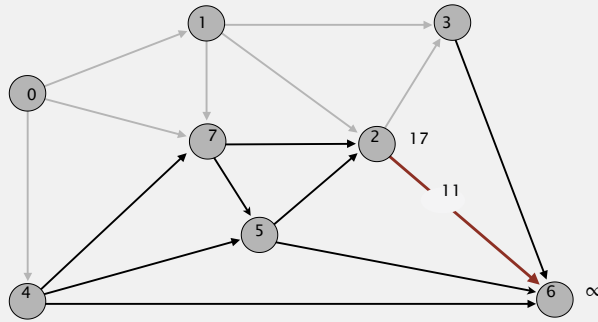
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

22

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

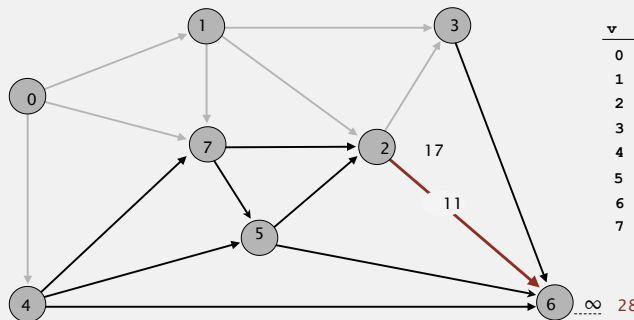
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2



23

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6	28.0	2→6
7	8.0	0→7

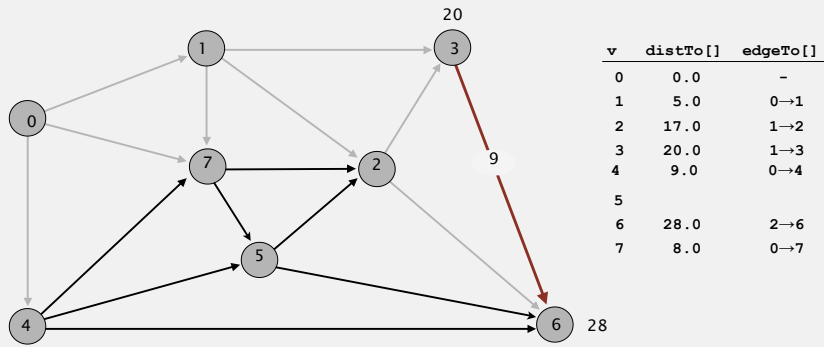
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2



24

Bellman-Ford algorithm

Repeat V times: relax all E edges.

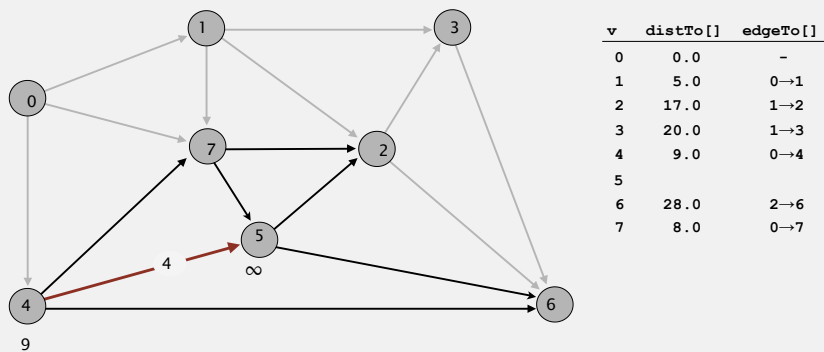
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

25

Bellman-Ford algorithm

Repeat V times: relax all E edges.

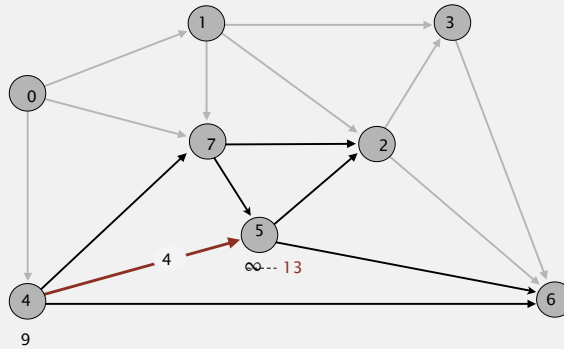
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

26

Bellman-Ford algorithm

Repeat V times: relax all E edges.

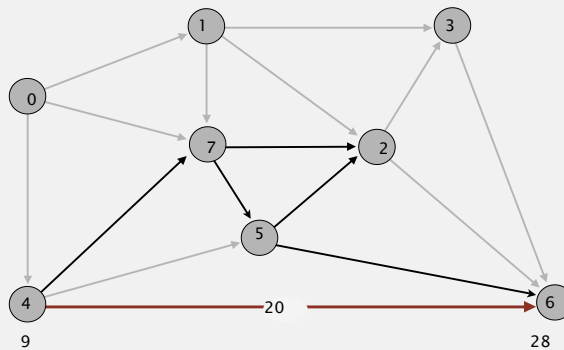
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

27

Bellman-Ford algorithm

Repeat V times: relax all E edges.

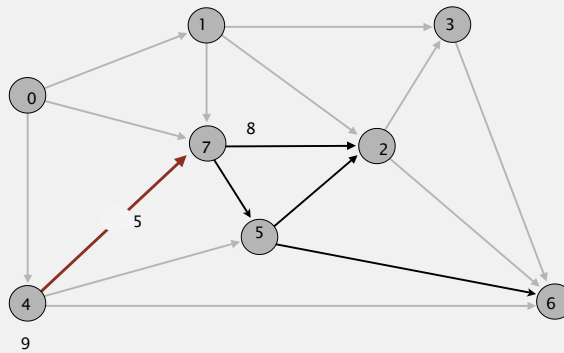
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

28

Bellman-Ford algorithm

Repeat V times: relax all E edges.

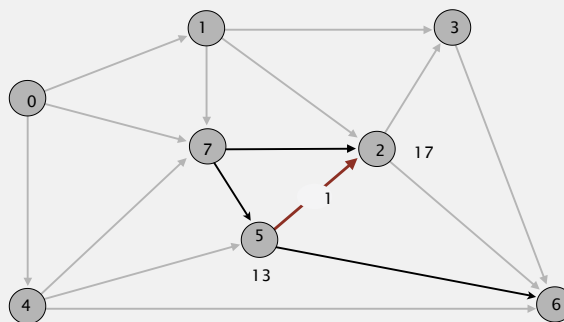
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

29

Bellman-Ford algorithm

Repeat V times: relax all E edges.

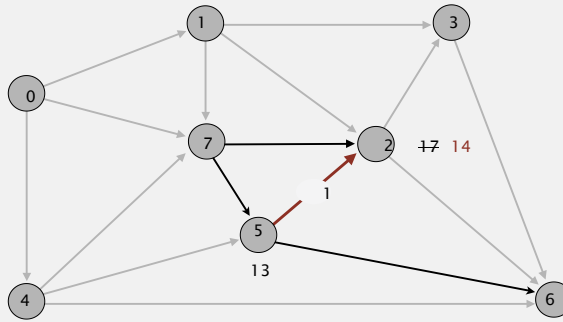
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

30

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

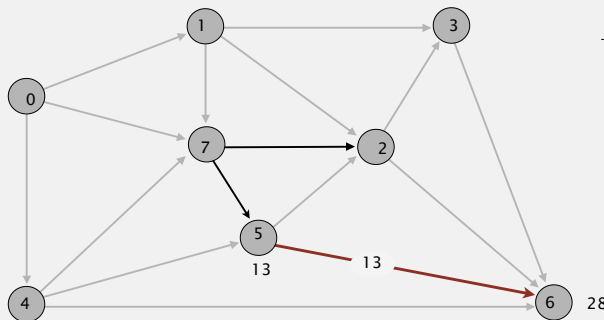
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

31

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	28.0	2→6
7	8.0	0→7

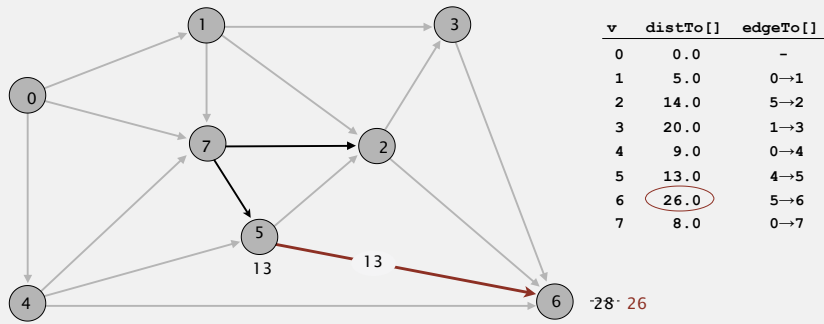
pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

32

Bellman-Ford algorithm

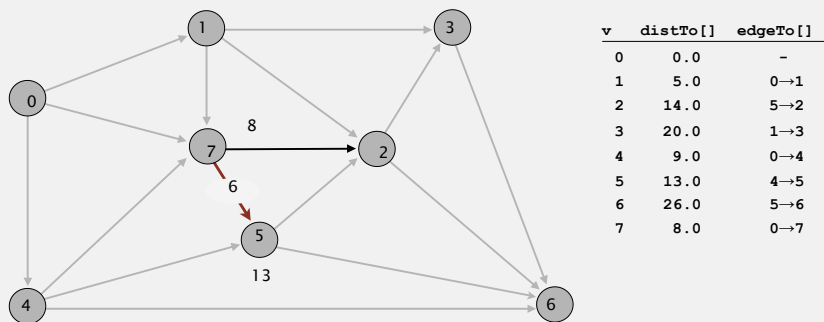
Repeat V times: relax all E edges.

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

33

Bellman-Ford algorithm

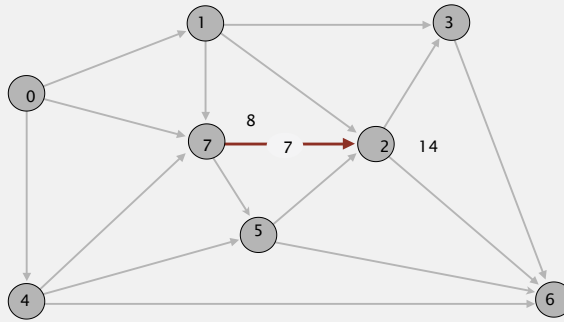
Repeat V times: relax all E edges.

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

34

Bellman-Ford algorithm

Repeat V times: relax all E edges.

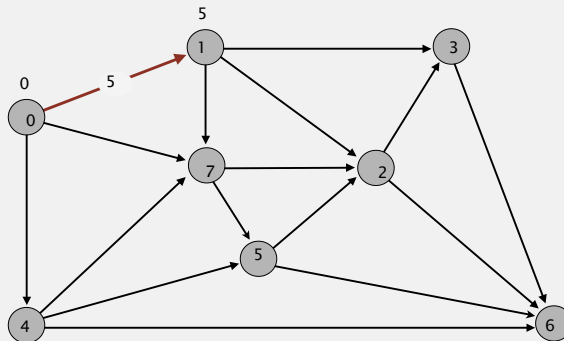
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0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 0

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

35

Bellman-Ford algorithm

Repeat V times: relax all E edges.

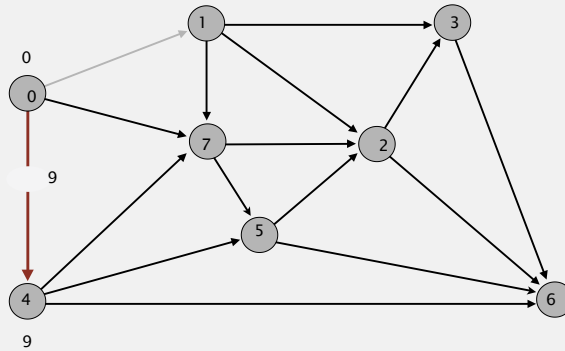
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0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

36

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

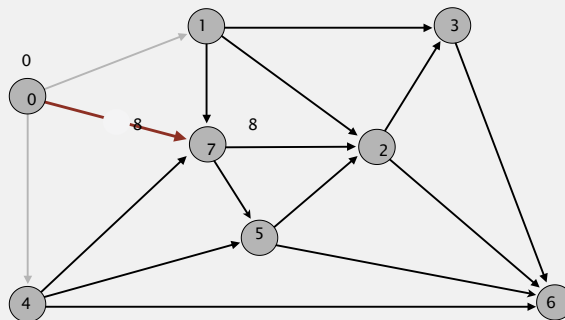
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

37

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

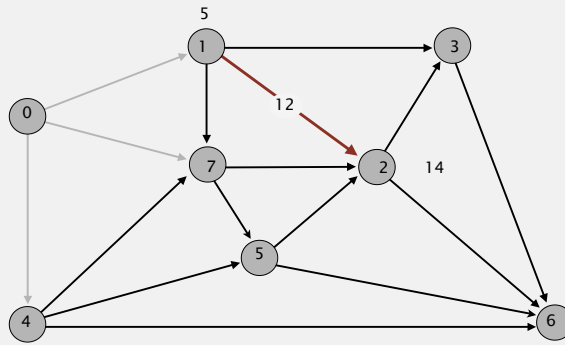
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

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Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

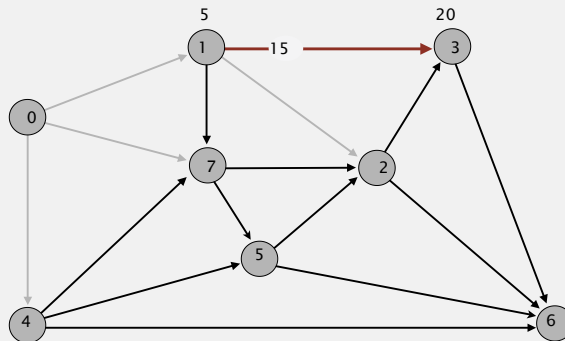
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

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Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

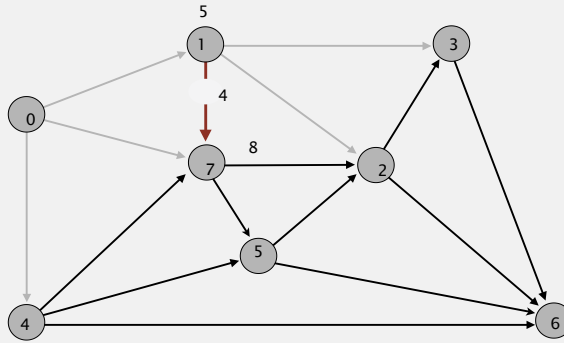
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

40

Bellman-Ford algorithm

Repeat V times: relax all E edges.

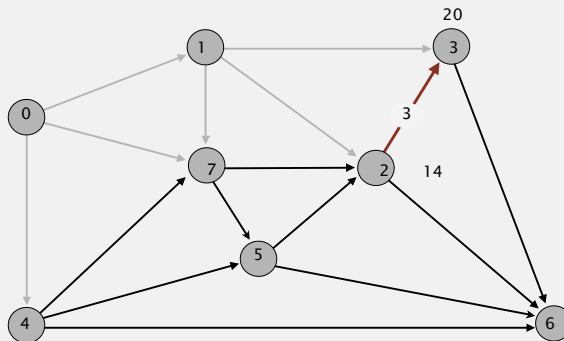
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

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Bellman-Ford algorithm

Repeat V times: relax all E edges.

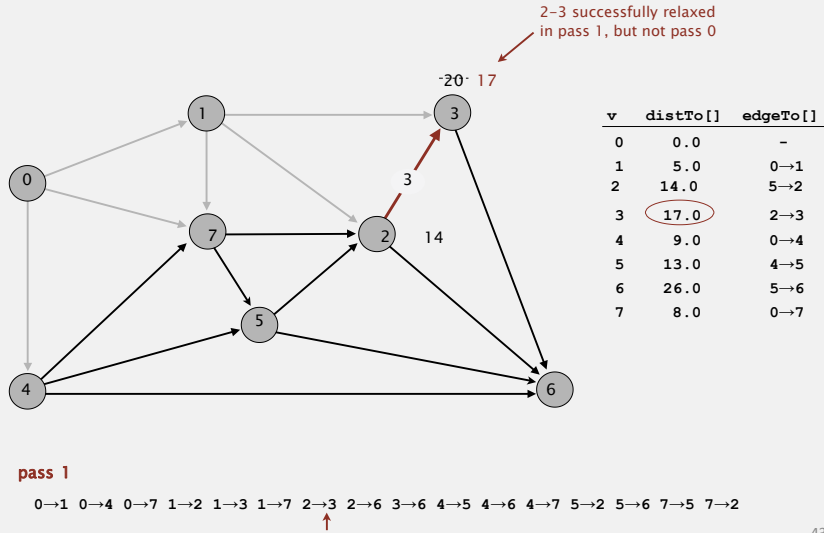
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

pass 1

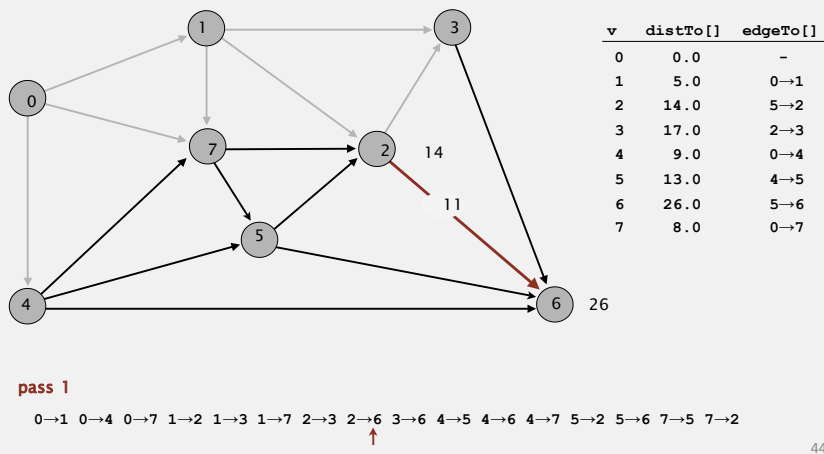
0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

42

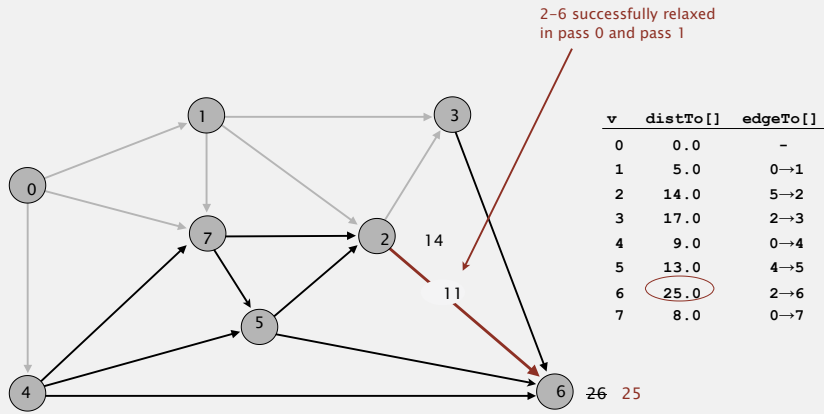
Bellman-Ford algorithm

Repeat V times: relax all E edges.

Bellman-Ford algorithm

Repeat V times: relax all E edges.

Bellman-Ford algorithm

Repeat V times: relax all E edges.

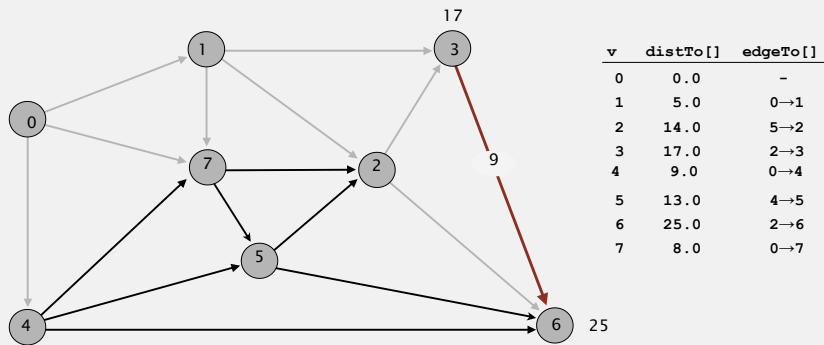
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

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Bellman-Ford algorithm

Repeat V times: relax all E edges.

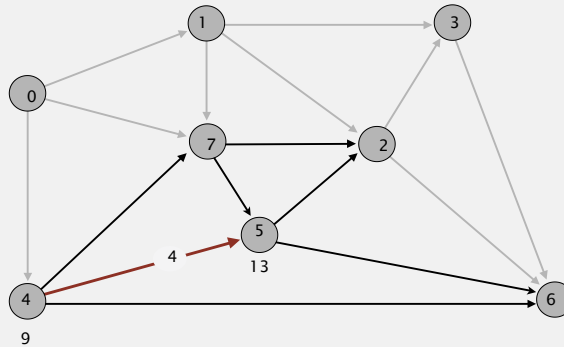
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

46

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

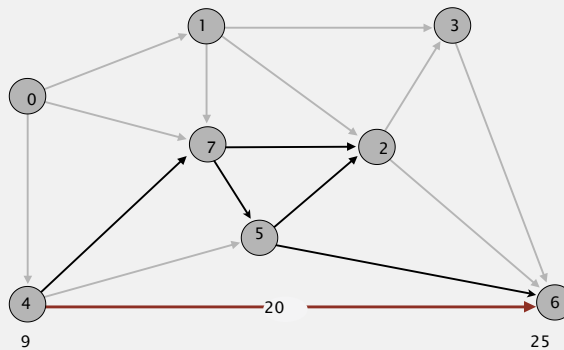
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

47

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

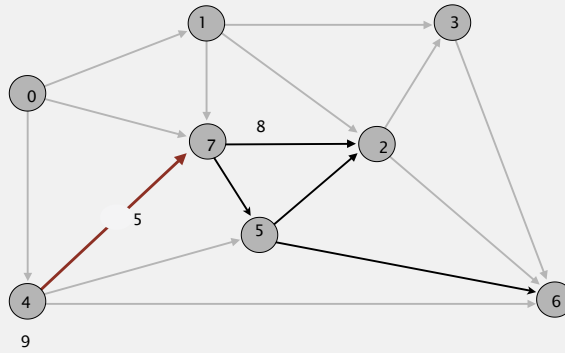
pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

↑

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Bellman-Ford algorithm

Repeat V times: relax all E edges.

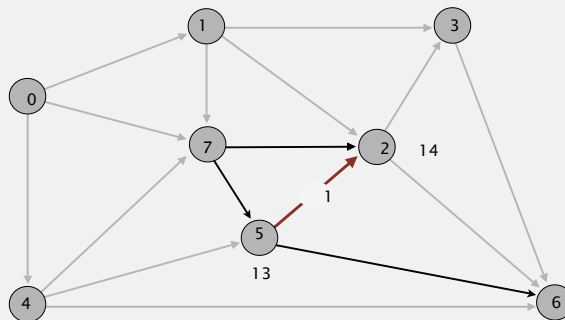
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

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Bellman-Ford algorithm

Repeat V times: relax all E edges.

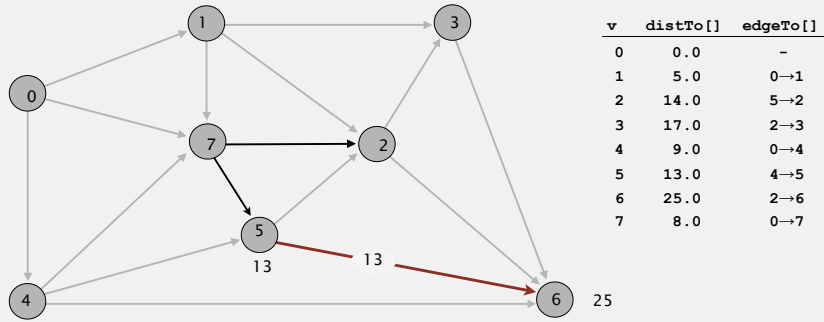
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

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Bellman-Ford algorithm

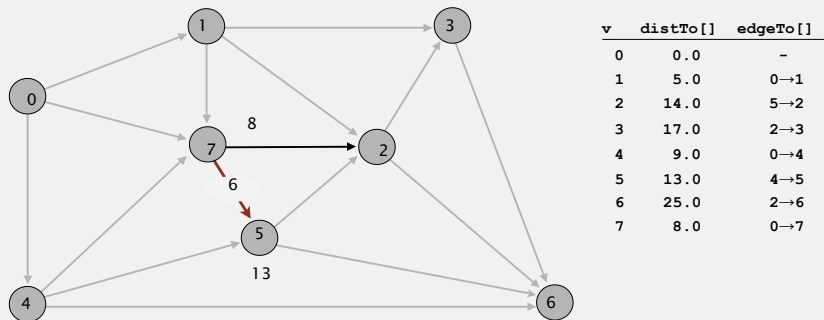
Repeat V times: relax all E edges.

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

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Bellman-Ford algorithm

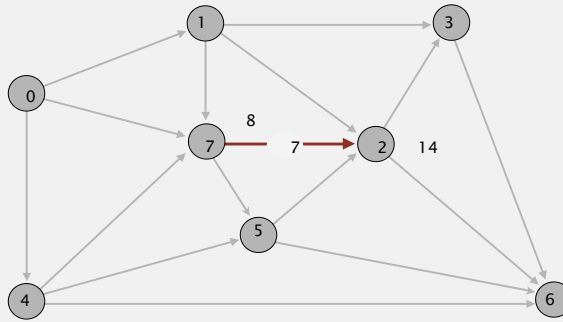
Repeat V times: relax all E edges.

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

52

Bellman-Ford algorithm

Repeat V times: relax all E edges.

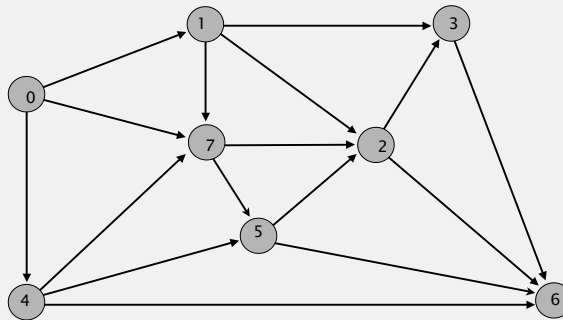
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

pass 1

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

53

Bellman-Ford algorithm

Repeat V times: relax all E edges.

v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

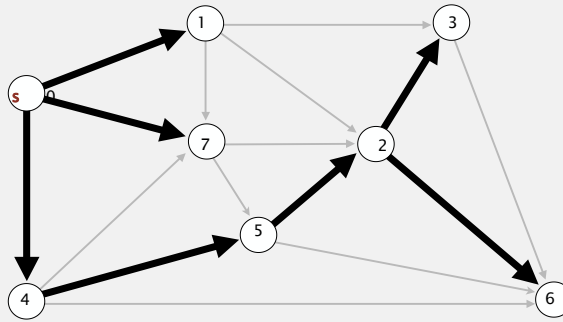
pass 2, 3, 4, ... (no further changes)

0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→5 7→2

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Bellman-Ford algorithm

Repeat V times: relax all E edges.

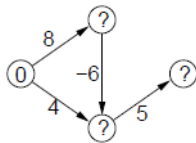


v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

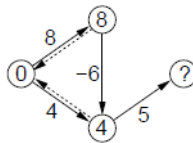
shortest-paths tree from vertex s

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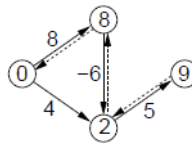
Bellman-Ford Algorithm Example



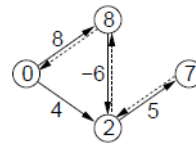
Initial configuration



After 1st relaxation phase



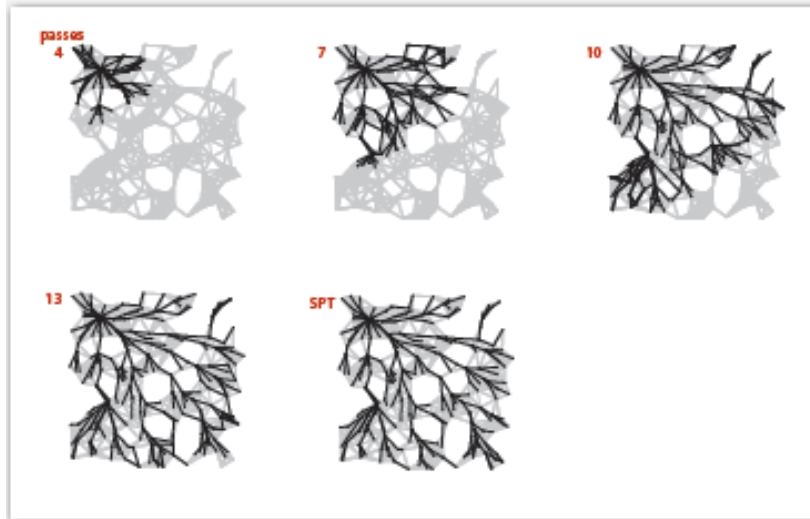
After 2nd relaxation phase



After 3rd relaxation phase

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Bellman-Ford algorithm visualization



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Bellman-Ford Algorithm: Correctness

- Shortest path will never visit the same vertex twice, so we know $k \leq V-1$
- Path consists of at most $V-1$ edges
- Let $\delta(s, v_i)$ be the true shortest path we have from s to v_i

$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i).$$

- After the i th pass of the “for- i ” loop, $d[v_i] = \delta(s, v_i)$
- Proof by induction:
 - Base Case: $d[v_1] = d[s] = 0$
 - Prior to i th pass: $d[v_{i-1}] = \delta(s, v_{i-1})$
 - After the i th pass, we have done relaxation on the edge (v_{i-1}, v_i)
 - Therefore, after the i th pass we have:

$$d[v_i] \leq d[v_{i-1}] + w(v_{i-1}, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i) = \delta(s, v_i).$$
 - Recall from Dijkstra’s algorithm that $d[v_i]$ is never less than $\delta(s, v_i)$
 - Therefore, $d[v_i] = \delta(s, v_i)$

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Bellman-Ford: Practical Improvement

Observation: If $\text{distTo}[v]$ does not change during pass i , no need to relax any edge pointing to v in pass $i+1$

FIFO implementation: Maintain **queue** of vertices whose distTo changed

↑
Be careful to keep at most one copy of each vertex on queue (why?)

Overall effect:

- The running time is still proportional to $E \times V$ in worst case
- But much faster than that in practice

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Single source shortest-paths implementation: cost summary

algorithm	restriction	typical case	worst case	extra space
topological sort	no directed cycles	$E + V$	$E + V$	V
Dijkstra (binary heap)	no negative weights	$E \log V$	$E \log V$	V
Bellman-Ford	no negative cycles	$E V$	$E V$	V
Bellman-Ford (queue-based)		$E + V$	$E V$	V

Remark 1. Directed cycles make the problem harder.

Remark 2. Negative weights make the problem harder.

Remark 3. Negative cycles makes the problem intractable.

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Finding a negative cycle

Observation. If there is a negative cycle, Bellman-Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.



Proposition. If any vertex v is updated in phase V , there exists a negative cycle (and can trace back `edgeTo[v]` entries to find it).

In practice. Check for negative cycles more frequently.

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