COMP 355
Advanced Algorithms

More on Network Flows
Section 7.1-7.3, 7.5-7.6 (KT)

Augmenting Path Algorithm

```
Augment(f, c, P) {
   b ← bottleneck(P)
   foreach e ∈ P {
      if (e ∈ E) f(e) ← f(e) + b        forward edge
      else       f(e^R) ← f(e) - b      reverse edge
   }
   return f
}
```

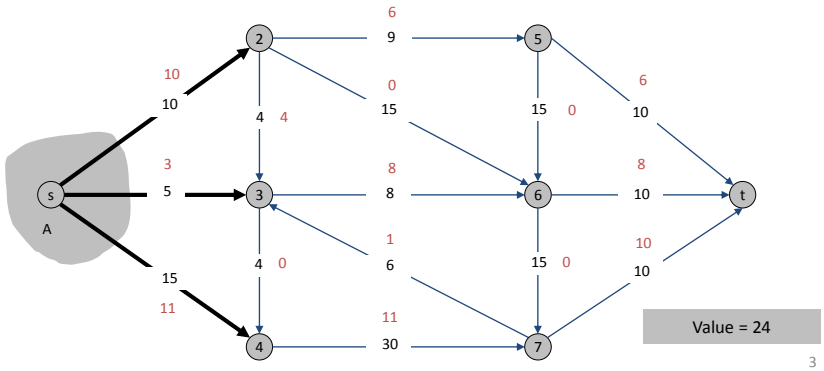
```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   G_f ← residual graph

   while (there exists augmenting path P) {
      f ← Augment(f, c, P)
      update G_f
   }
   return f
}
```

Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) \; - \; \sum_{e \text{ in to A}} f(e) \; = \; v(f)$$



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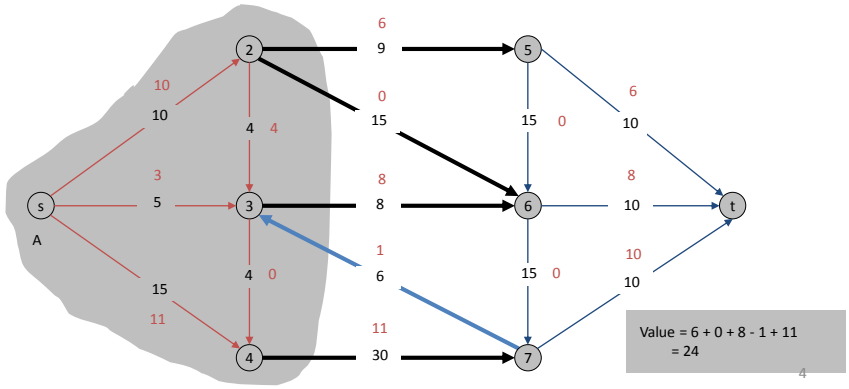

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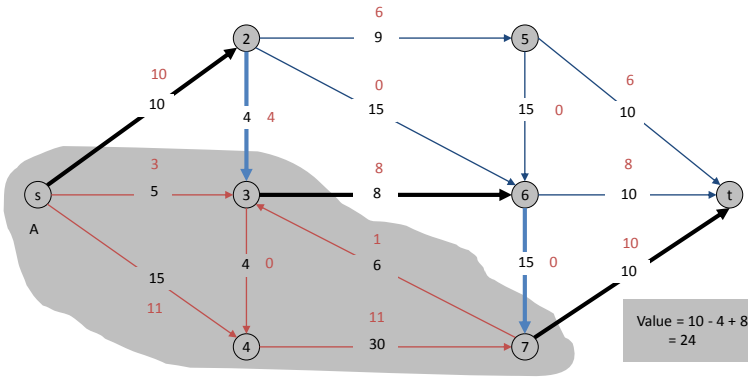


Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

by flow conservation, all terms except v = s are 0

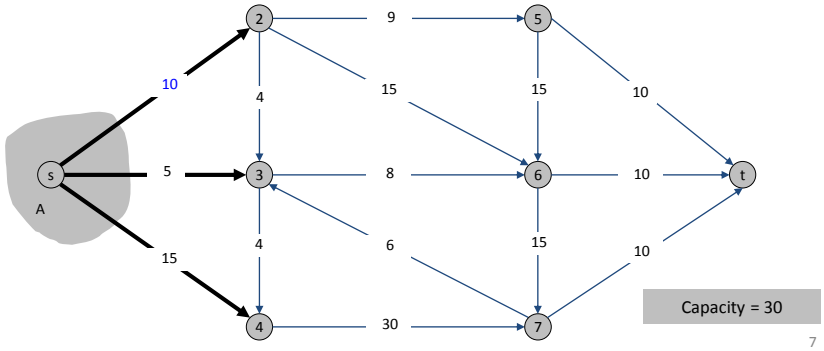
$$\longrightarrow \quad = \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e).$$

Flows and Cuts

Weak duality. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

Cut capacity = 30 ⇒ Flow value ≤ 30



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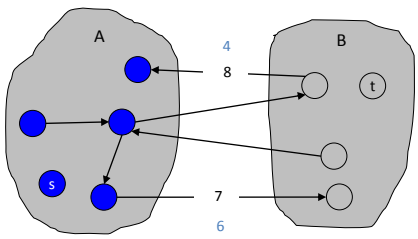
Flows and Cuts

Weak duality. Let f be any flow. Then, for any s-t cut (A, B) we have v(f) ≤ cap(A, B).

Pf.

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
&\le \sum_{e \text{ out of } A} f(e) \\
&\le \sum_{e \text{ out of } A} c(e) \\
&= \text{cap}(A, B)
\end{aligned}
$$



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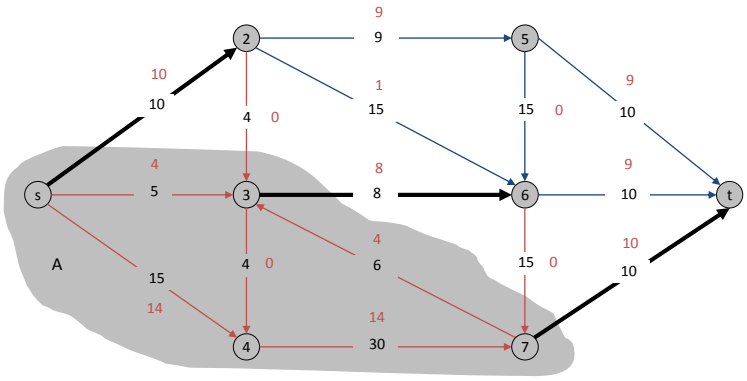
Certificate of Optimality

Max-Flow/Min-Cut Theorem.
Let f be any flow, and let (A, B) be any cut.
If v(f) = cap(A, B), then f is a max flow and (A, B) is a min cut.

Value of flow = 28
Cut capacity = 28 $\Rightarrow$ Flow value $\leq$ 28



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Max-Flow Min-Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Ford-Fulkerson 1956] The value of the max flow is equal to the value of the min cut.

Proof strategy. We prove both simultaneously by showing the following are equivalent.

(i) There exists a cut (A, B) such that v(f) = cap(A, B).

(ii) Flow f is a max flow.

(iii) There is no augmenting path relative to f.

(i) $\Rightarrow$ (ii) This was the corollary to weak duality lemma.

(ii) $\Rightarrow$ (iii) We show contrapositive.

- Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.

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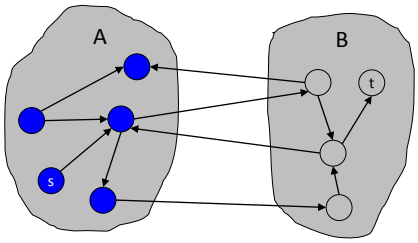
Proof of Max-Flow Min-Cut Theorem

(iii) $\Rightarrow$ (i)

- Let f be a flow with no augmenting paths.
- Let A be set of vertices reachable from s in residual graph.
- By definition of A, s $\in$ A.
- By definition of f, t $\notin$ A.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e) \\ &= \sum_{e \text{ out of } A} c(e) \\ &= cap(A, B) \quad \square \end{aligned}$$



original network

Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value f(e) and every residual capacity $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \leq mC$ iterations.
Pf. Each augmentation increases value by at least 1. ▪

Corollary. If C = 1, Ford-Fulkerson runs in O(mn) time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer.

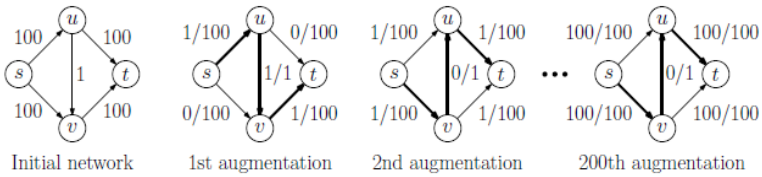
Pf. Since algorithm terminates, theorem follows from invariant. ▪

Ford-Fulkerson: Exponential Number of Augmentations

Q. Is generic Ford-Fulkerson algorithm polynomial in input size?

m, n, and log C

A. No. If max capacity is C, then algorithm can take C iterations.



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Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- If capacities are irrational, algorithm not guaranteed to terminate!

Goal: choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]

- Max bottleneck capacity.
- Sufficiently large bottleneck capacity.
- Fewest number of edges.

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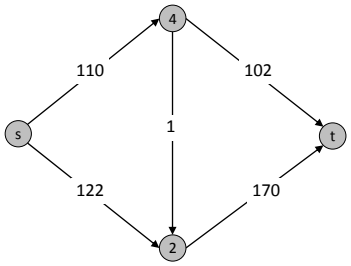
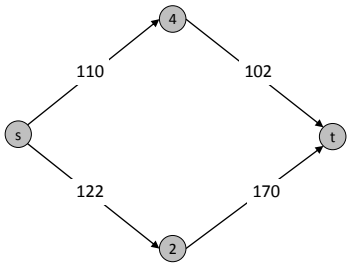
Capacity Scaling

Intuition. Choosing path with highest bottleneck capacity increases flow by max possible amount.

- The sum of capacities of the edges leaving s is $C = \sum_{(s,v) \in E} c(s, v)$
- Define Δ to be the largest power of 2, such that $\Delta \leq C$
- Let $G_f(\Delta)$ be the subgraph of the residual graph consisting of only arcs with capacity at least Δ.



G_f



$G_f(100)$

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Capacity Scaling

```
Scaling-Max-Flow(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   Δ ← smallest power of 2 greater than or equal to C
   G_f ← residual graph

   while (Δ ≥ 1) {
      G_f(Δ) ← Δ-residual graph
      while (there exists augmenting path P in G_f(Δ)) {
         f ← augment(f, c, P)
         update G_f(Δ)
      }
      Δ ← Δ / 2
   }
   return f
}
```

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Capacity Scaling: Correctness

Assumption. All edge capacities are integers between 1 and C.

Integrality invariant. All flow and residual capacity values are integral.

Correctness. If the algorithm terminates, then f is a max flow.

Pf.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths. ▪

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Capacity Scaling: Running Time

Lemma 1. The outer while loop repeats $1 + \lceil \log_2 C \rceil$ times.

Pf. Initially $C \leq \Delta < 2C$. Δ decreases by a factor of 2 each iteration. ▪

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then the value of the maximum flow is at most $v(f) + m\,\Delta$. ← proof on next slide

Lemma 3. There are at most 2m augmentations per scaling phase.

- Let f be the flow at the end of the previous scaling phase.
- L2 $\Rightarrow$ $v(f^*) \leq v(f) + m\,(2\Delta)$.
- Each augmentation in a Δ-phase increases v(f) by at least Δ. ▪

Theorem. The scaling max-flow algorithm finds a max flow in O(m log C) augmentations. It can be implemented to run in $O(m^2 \log C)$ time. ▪

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Capacity Scaling: Running Time

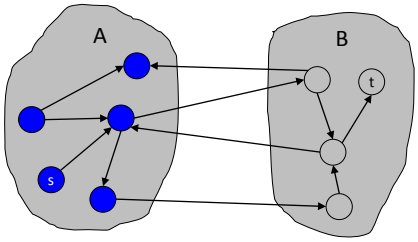
Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then value of the maximum flow is at most v(f) + m Δ.

Pf. (almost identical to proof of max-flow min-cut theorem)

- We show that at the end of a Δ-phase, there exists a cut (A, B) such that cap(A, B) $\leq$ v(f) + m Δ.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of A, s $\in$ A.
- By definition of f, t $\notin$ A.

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e) \\
&\geq \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\
&= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\
&\geq cap(A, B) - m\Delta
\end{aligned}
$$

□



original network

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Edmonds-Karp Algorithm

- Neither of the algorithms we have seen so far runs in "truly" polynomial time
- Edmonds and Karp developed the first polynomial-time algorithm for flow networks.
  - Uses Ford-Fulkerson as basis
  - Modification: when finding the augmenting path, we compute the s-t path in the residual network having the smallest number of edges
    - Note that this can be accomplished by using BFS to compute the augmenting path
  - It can be shown that the total number of augmenting steps using this method is O(nm) (Proof in CLRS)
  - Overall runtime = $O(nm^2)$

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Other Algorithms

- KT discusses pre-flow push algorithm
  - Number of variants of this algorithm
  - Simplest version runs in $O(n^3)$ time
- Another quite sophisticated algorithm runs in time $O(\min(n^{2/3}, m^{1/2})m \log n \log U)$, where U is an upper bound on the largest capacity.

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Practice

Professor Adam has two children, who unfortunately, dislike each other. The problem is so severe that not only do they refuse to walk to school together, but in fact each one refuses to walk on any block that the other child has stepped on that day. The children have no problem with their paths crossing at a corner. Fortunately, both the professor's house and the school are on corners, but beyond that he is not sure if it is going to be possible to send both of his children to the same school. The professor has a map of the town. Show how to formulate the problem of determining whether both his children can go to the same school as a maximum-flow problem.

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