Homework 5: Network Flows & NP-Completeness

Handed out Wednesday, November 8. Due at the start of class Monday, November 20.

Problem 1. Present an efficient algorithm for the following problem. Given a directed graph $G = (V, E)$ and two vertices s and t, determine the maximum number of paths between s and t, such that, other than s and t, these paths do not share any vertices in common. (In Fig. 1, there are 4 such paths.)

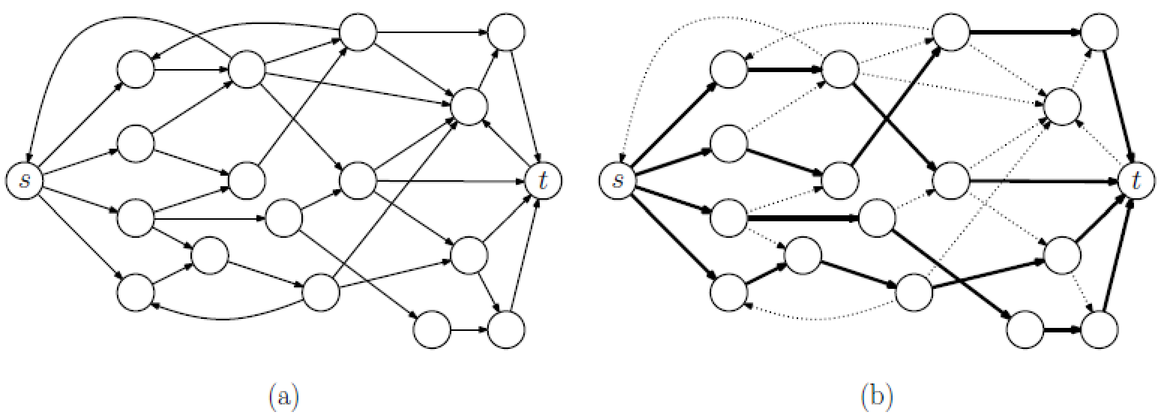

Figure 1: Problem 1.

(Hint: As a start, devise a way to compute network flows such that the total flow through any vertex is at most 1.)

Problem 2. Highly intelligent aliens from another world come to Earth and tell us (1) that the 3-Coloring problem (which is NP-complete) is solvable in $O(n^9)$ time, and (2) there is no algorithm for 3-Coloring that runs faster than $\Omega(n^7)$ time in the worst case. (Here n denotes the number of nodes in the graph.)

For each of the following assertions, indicate whether it follows from the information the aliens have given us. Also, provide a short explanation in each case.

(i) All NP-complete problems are solvable in polynomial time.

(ii) All problems in NP, even those that are not NP-complete, are solvable in polynomial time.

(iii) All NP-hard problems are solvable in polynomial time.

(iv) All NP-complete problems are solvable in $O(n^9)$ time.

(v) No NP-complete problem can be solved faster than $\Omega(n^7)$ time in the worst case.

(vi) There is no algorithm for the 4-Coloring problem that runs in time faster than $\Omega(n^7)$ time.

Problem 3. Most NP-complete problems are expressed as decision problems, where the answer is simply "yes" or "no," but in practice a user wants to know why the answer is "yes" or "no." In this problem, we will show that if we are given access to an algorithm for a decision problem, it is often possible to use this to obtain the entire answer.

(a) **Hamiltonian Cycle**: Given an undirected graph $G = (V, E)$, does there exist a cycle that visits every vertex of graph exactly once?
Suppose that we had a function `Hamiltonian Cycle(G)`, which (by some miracle) ran in polynomial time and returns true if G has a Hamiltonian cycle and false otherwise. Show that if G has a Hamiltonian cycle, then it is possible to use this function (as a black box) to compute the sequence of vertices on the Hamiltonian cycle in polynomial time.
(Let $n = |V|$ and $m = |E|$. For full credit, solve this problem using $O(m)$ calls to the function. For partial credit, any polynomial number of calls is allowed.)

(b) **3-Colorable**: Given an undirected graph $G = (V, E)$ can the vertices of G be labeled with three colors (say, 1, 2, and 3) such that no edge is incident to two vertices of the same color?
Suppose that we had a function `Three Color(G)`, which (by some miracle) ran in polynomial time and returns true if G is 3-colorable and false otherwise. Show that if G is 3-colorable, then it is possible to use this function (as a black box) to determine the assignment of colors to the vertices. (Let $n = |V|$ and $m = |E|$. For full credit, solve this problem using $O(n)$ calls to the function. For partial credit, any polynomial number of calls is allowed.)

Problem 4. In the High-Degree Independent Set (HDIS) problem, you are given an undirected graph $G = (V, E)$ and an integer k, and you want to know whether there exists an independent set V' in G of size k such that each vertex of V' is of degree at least k. (For example, the graph in Fig. 2 has an HDIS for $k = 3$, shown as the shaded vertices. Note that it does not have an HDIS for $k = 4$. Although adding the topmost vertex would still yield an independent set, this vertex does not have degree at least four.)

(a) Show that HDIS is in NP.

(b) Show that HDIS is NP-hard. (Hint: Reduction from the standard independent set problem (IS).)

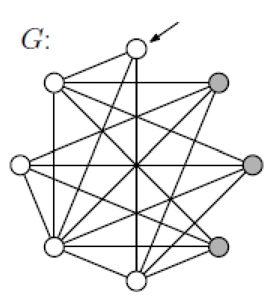

Figure 2: Problem 4. High-degree independent set.